Local average treatment effects for multiple treatments with multiple instruments*

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Abstract

When individuals self-select into multiple treatments, additional assumptions are needed to identify treatment effects with instrumental variables. Suppose each treatment is targeted by a single instrument, and one of three equivalent assumptions holds: no defiers, identical compliers, or additive random utility. With continuous instruments, average marginal treatment effects for individuals indifferent between control and K treatments ("MTE-K") are identified. With price instruments in a 3x3 experimental design, absent income effects, local average treatment effects for two treatments for common compliers ("LATE-2") are informatively partially identified: as experimental price variation shrinks, LATE-2 bounds converge to MTE-2.

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When individuals self-select into treatments, under what conditions are treatment effects identified? With a single treatment, a weighted average of treatment effects is identified under an economically interpretable "monotonicity" assumption on an instrumental variable (Imbens & Angrist, 1994; Heckman & Vytlacil, 2005): increases in the instrument induce some individuals to shift from treatment to control, but do not induce any to shift from control to treatment. In contrast, with multiple treatments, the widely applied result with a single treatment that two-stage least squares estimates a local average treatment effect no longer holds, even when there is at least one instrument per treatment — additional assumptions are then necessary to identify treatment effects (Behaghel et al., 2013; Kirkeboen et al., 2016).

In this paper, I establish that with two treatments, a local average treatment effect for both treatments for a common set of compliers ("LATE-2") is informatively partially identified in a 3x3 experimental design that cross-randomizes zero, one, and two unit increases in the price of each treatment. By placing strong restrictions on the instruments (cross-randomized prices), albeit one under the control of the researcher, I enable informative partial identification under a simple and economically interpretable assumption: there are no income effects on treatment demand.

The intuition underlying partial identification of LATE-2 with cross-randomized prices is as follows. An increase in the price of choice $d \in \{a, b\}$ pushes individuals into an outside option 0 who are almost indifferent between 0 and d. A simultaneous increase in the price of both a and b, relative to increasing both prices separately, additionally pushes individuals into 0 who are almost indifferent between 0, a, and b; this difference-in-differences therefore enables identification of the average outcome under 0 of these almost indifferent individuals. Further, assuming no income effects, we can construct a *synthetic* increase in the price of 0 from an equal decrease in the price of both a and b; just as for 0, we can then estimate the average outcome under either a or b for individuals almost indifferent between 0, a, and b. Unfortunately, the three groups of almost indifferent individuals (those for whom average outcomes under 0, under a, and under b can be estimated) are not identical, due to the discreteness of the price changes. I therefore propose bounds on mean potential outcomes for the intersection of these three groups, which in turn bound LATE-2.

In Section 1, I formalize assumptions on selection of individuals into multiple treatments, and on outcomes. In subsequent sections, I apply these assumptions to define and identify the marginal treatment effect ("MTE") for individuals indifferent between a set of treatments with continuous variation in instruments, and to establish partial identification of LATE-2 in the 3x3 experimental design.

As in Mountjoy (2022), I maintain throughout that responses to changes in the value of the instruments satisfy unordered partial monotonicity ("UPM"), that each instrument shifts agents towards the instrument's associated choices and away from other choices. I show this assumption is equivalent to the random utility model (McFadden, 1981): individuals make utility maximizing choices when selecting into treatment, and each instrument, as if a price, increases individual utilities associated with its targeted choice.

I consider three additional assumptions that restrict the heterogeneity of treatment responses to changes in the value of the instruments across individuals. Existing work has developed approaches to identification of treatment effect parameters related to LATE-2 and MTE, and falsification tests under each assumption.

- No defiers ("ND") imposes a global notion of monotonicity: for any change in the value of the instruments, no two individuals have opposing treatment responses (Navjeevan & Pinto, 2022).
- Identical compliers ("IC") imposes a marginal notion of monotonicity: the same individuals are marginal between a pair of treatments, whether revealed by small changes in either instrument targeting each treatment.[1]
- The Additive Random Utility Model ("ARUM") imposes that agent utility is additively separable in unobserved heterogeneity and the instrument (Lee & Salanié, 2018).

In Section 2, I show that conditional on UPM, and additional technical assumptions, ND, IC, and ARUM are equivalent. This equivalence extends a result from Vytlacil (2002), that ND and ARUM are equivalent when treatment is binary, to the setting with multiple unordered treatments. The proof builds closely on Mogstad et al. (2021), who show that with multiple instruments and a binary treatment, ND is effectively equivalent to homogeneous instrument sensitivity. Under additional technical assumptions, homogeneous instrument sensitivity is equivalent to ARUM. I

[1] IC implies the closely related assumption of comparable compliers from Mountjoy (2022), which imposes the weaker restriction that individuals with the same mean potential outcomes are marginal. I discuss the distinction between assumptions on selection into treatment, on which this paper focuses, and assumptions on outcomes, which may enable identification and need not imply assumptions on selection, in Section 1.2.

apply their approach and extend it to IC – ND and IC are equivalent to homogeneous instrument sensitivity, and therefore ARUM.

As a consequence, assuming UPM and either ND or IC is equivalent to assuming that instruments are prices and choices satisfy ARUM.

In Section 3, I show that with continuous variation in prices under ARUM, the average MTE across individuals who are indifferent between any set of treatments is identified. When utility indices are known, identification of MTE for individuals indifferent between all treatments is a corollary of Theorem 3.1 of Lee & Salanié (2018). Absent income effects, utility indices are prices and are therefore observed; with income effects, identification of utility indices with continuous variation in prices has been established by Allen & Rehbeck (2019) and Bhattacharya (2023). While these are known results, combining them to establish local identification of MTE across individuals indifferent between all treatments with unknown utility indices is novel. Complementarily, I extend known results to identification of the average MTE across individuals who are indifferent between any set of treatments, bridging identification results from Lee & Salanié (2018) (which apply to indifference between all treatments) and Mountjoy (2022) (which apply to indifference between a pair of treatments).

In many empirical applications, researchers do not have access to continuous variation in multiple instruments; in Section 4, I therefore derive bounds on LATE-2 with discrete variation in prices when there are no income effects on treatment demand. I do so in the 3x3 experimental design that cross-randomizes zero, one, and two unit price increases for each of two treatments a and b relative to control 0. The derivation proceeds in 4 steps:

- I characterize all 19 treatment response types and establish identification of their probabilities. These include 8 "complier" groups who choose each of 0, a, and b at one or more assigned prices.
- I show that mean potential outcomes for selected pairs of complier groups are equal to the ratio of difference-in-difference estimands. This enables partial identification of mean potential outcomes under 0, a, or b for selected complier groups.
- I define LATE-2 as the average treatment effect across 2 (of the 8) complier groups: these are the intersection of the complier groups for whom the mean potential outcome under 0, under a, and under b is partially identified.

- Absent additional assumptions, I show that LATE-2 bounds (Manski, 1990) are, in general, *uninformative* in the limit as price increases approach 0. As an alternative, I derive LATE-2 bounds under a generalization of monotone treatment selection (Manski & Pepper, 2000), monotonicity of local average treatment response ("MLATR"). I show that MLATR holds to first order with respect to the size of price increases, and correspondingly that LATE-2 bounds under MLATR converge to MTE in the limit as the price increases approach 0.

When instruments are prices and there are no income effects, this paper proposes LATE-2 bounds under MLATR as the natural extension of LATE (Imbens & Angrist, 1994) to the setting with two treatments, and the 3x3 experimental design as the associated extension of an experimental subsidy. When individuals select into a single treatment, as if maximizing utility, the "encouragement design" that randomizes a subsidy for treatment enables identification of the average treatment effect for compliers. This paper establishes that for any pair of treatments, an encouragement design that cross-randomizes subsidies for both treatments enables informative partial identification of both average treatment effects for a common set of compliers (LATE-2) under the additional assumption of no income effects. Further, just as LATE converges to MTE as the price increase approaches 0, LATE-2 bounds under MLATR converge to MTE.

Examples Partial identification of LATE-2, and the associated proposed 3x3 experimental design, are particularly relevant when multidimensional selection bias is plausible or of interest. Example applications include:

- Agents select from a menu of discrete choices, and choices are directly influenced by the price and multidimensional impacts of each choice, as in the choice of school (Mountjoy, 2022) or residence (Bergman et al., 2019; Pinto, 2022; Agness & Getahun, 2024).
- Agents select from a menu of discrete choices with not-fully internalized consequences which are plausibly correlated with willingness-to-pay, as is the case with inattention (Allcott & Taubinsky, 2015, to costs for lightbulbs) or externalities (Berkouwer & Dean, 2022, from pollution for cookstoves).
- Agents make ordered choices and both the level and elasticity of demand may correlate with the schedule of impacts, as for input misallocation across farms

(Christian et al., 2023) or health insurance enrollment (Rose & Shem-Tov, 2023).

Literature review This paper contributes to a growing literature that analyzes identification of marginal and local average treatment effects with multiple treatments, extending the single treatment analysis of Heckman & Vytlacil (2005) and Imbens & Angrist (1994) respectively.

This paper is most closely related to recent work analyzing identification of treatment effects under UPM and ARUM. Under a generalization of ARUM, with known utility indices, Lee & Salanié (2018) establish local identification of MTE; known utility indices is equivalent to assuming no income effects as, absent income effects, prices are utility indices. Under UPM and an assumption related to IC, with unknown utility indices, Mountjoy (2022) establishes local identification of the average MTE among individuals indifferent between a pair of treatments. In this paper, I clarify links across these results by showing IC and ARUM are equivalent conditional on UPM, and I extend them by establishing local identification, with unknown utility indices, of average MTE among individuals indifferent across any set of treatments.

Alternative treatment effect parameters have been proposed that can be identified with discrete instruments. With one instrument that shifts individuals from 0 to either a or b, probabilities of all treatment response types are identified as are many of their potential outcome distributions, with the crucial exception that the distribution of outcomes under 0 for 0-to-a and 0-to-b compliers cannot be separated (Kline & Walters, 2016; Feller et al., 2016; Rose & Shem-Tov, 2023). In more general designs, when instruments act as if prices, economic theory can often impose restrictions on choices that enable identification of complier group probabilities (Kline & Tartari, 2016) and treatment effects (Heckman & Pinto, 2018; Pinto, 2022; Buchinsky et al., 2024) not otherwise identified: this paper extends this logic to enable partial identification of LATE-2 absent income effects, and identification of MTE.

Alternative assumptions have been suggested that achieve identification of MTE or LATE-2. These include large instrument support (Heckman et al., 2008), observing counterfactual choices (Kirkeboen et al., 2016), additive separability of exogenous characteristics in marginal treatment response when there is only one instrument for a or b (Kline & Walters, 2016; Feller et al., 2016; Hull, 2018), or access to an instrument for a or b and another for b holding fixed the choice of a or b, motivated by sequential rather than simultaneous treatment choice (Arteaga, 2023; Humphries et al., 2023; Kamat et al., 2024). This paper is distinct in proposing an experimental design (the 3x3 factorial design cross-randomizing prices) in order to enable partial identification of LATE-2 under the economically interpretable assumption of no income effects.

Structure of the paper Section 1 formalizes assumptions on selection into multiple treatments, and Section 2 establishes an equivalence relation across assumptions that restrict heterogeneity of treatment response. Section 3 establishes identification of MTE for multiple treatments with continuous variation in prices. Section 4 establishes partial identification of LATE-2 for a pair of treatments in a 3x3 experimental design. Section 5 concludes.

1 Modeling selection into multivalued treatment

This section describes a model of selection into multivalued treatment and associated assumptions. Section 1.1 describes the selection framework with K treatments and K instruments. Section 1.2 describes and interprets potential additional assumptions related to monotonicity on the selection model that enable identification of treatment effects. Section 1.3 interprets the relationship between these and other assumptions on the selection model through their restrictions on admissible treatment response graphs. Sections 1.4 and 1.5 develop additional technical assumptions on selection and assumptions on outcomes, respectively.

1.1 Selection into multivalued treatment

Let $\mathcal{I}$ denote the population of individuals $i \in \mathcal{I}$, with an associated probability measure $\mathbf{P}$ and expectation $\mathbf{E}$. Let $D_i(z) \in \mathcal{K} \equiv \{0, 1, \ldots, K\}$ denote the potential treatment status of individual i if their instrument $Z_i \equiv (Z_{i1}, \ldots, Z_{iK})$ were set to $z \equiv (z_1, \ldots, z_K)$, where Z_i and z have support on a K-dimensional finite interval $\mathcal{Z} \subset \mathbb{R}^K$. Further let $D_{id}(z) \equiv \mathbf{1}[D_i(z) = d]$ indicate that individual i would have treatment status d if their instrument were set to z.

Throughout the paper, I use $(z'_{\mathcal{K}_0}, z_{-\mathcal{K}_0})$ to denote the vector where, for each $k \in \mathcal{K}_0$, the kth element of z, z_k, is set equal to z'_k, noting that $z \equiv (z_{\mathcal{K}_0}, z_{-\mathcal{K}_0})$; when $\mathcal{K}_0$ is a singleton, that is $\mathcal{K}_0 = \{k\}$, I instead write (z_k, z_{-k}). Similarly, I use

$(z'_k, z'_\ell, z_{-\{k,\ell\}})$ to denote the vector where the kth and ℓth elements of z, z_k and z_ℓ, are set equal to z'_k and z'_ℓ, respectively, and note that $z \equiv (z_k, z_\ell, z_{-\{k,\ell\}})$.

In Section 4, and at times throughout this paper, I restrict to the case when $K = 2$. I then follow Heckman & Pinto (2018) and let $\mathcal{K} \equiv \{0, a, b\}$ to emphasize both that a and b are not necessarily ordered, and that $K = 2$.

Throughout this section, I interpret this setup as if $\mathcal{K}$ consists of a default option 0 and K mutually exclusive products $\{1, \ldots, K\}$, one of which the household chooses to consume. Through this lens, I interpret the kth dimension of the instrument, Z_{ik}, as the price of product k assigned to individual i.

This model, and assumptions on selection made in this section, are closely related to assumptions in Mountjoy (2022) and I at times use their context for concreteness. In their empirical setting, $\mathcal{K} \equiv \{0, 2, 4\}$, and the values of treatment statuses correspond to not attending college, initially attending two-year college, and initially attending four-year college, respectively. In their application, the realizations of the instrument Z_{i2} and Z_{i4} correspond to distance from i to the nearest two-year college and to the nearest four-year college, respectively.

1.2 Monotonicity

With a binary treatment and scalar instrument, the assumption that treatment status is increasing in the value of the instrument ("monotonicity", in Imbens & Angrist, 1994) implies no defiers; in contrast, with multiple unordered treatments, common restrictions on the sign and heterogeneity of treatment responses are no longer nested. Section 1.2.1 describes an assumption that restricts the sign of treatment responses to changes in the value of each instrument, unordered partial monotonicity, and links it to a random utility model. Section 1.2.2 describes assumptions that restrict heterogeneity in responses across individuals: identical compliers, no defiers, and an additive random utility model.

1.2.1 Unordered partial monotonicity and the random utility model

In many applications, the instrument z acts similarly to (or is) the vector of prices of the treatments. When the price of a given treatment increases, the canonical random utility model restricts the potential treatment responses: marginal individuals shift out of the now more expensive treatment into other treatments, but no individuals are induced to shift between treatments that did not experience price changes. Mountjoy (2022) formalizes this assumption with two treatments; I restate it for K treatments with the notation in Section 1.1.

Assumption UPM (Unordered Partial Monotonicity). *For all $i \in \mathcal{I}$, $k \in \mathcal{K} \setminus \{0\}$, and $(z_k, z_{-k}), (z'_k, z_{-k}) \in \mathcal{Z}$ with $z'_k < z_k$,*

$$D_{ik}(z'_k, z_{-k}) \geq D_{ik}(z_k, z_{-k})$$

and for all $\ell \in \mathcal{K} \setminus \{k\}$,

$$D_{i\ell}(z'_k, z_{-k}) \leq D_{i\ell}(z_k, z_{-k})$$

In the context of college attendance choices in Mountjoy (2022), Assumption UPM imposes the following restrictions on treatment status:

- Decreasing the distance to the nearest two-year college causes individuals to shift from not attending college to two-year college, and to shift from attending four-year college to attending two-year college. It does not cause any individuals to stop attending two-year college, or to shift between not attending college and four-year college.
- Similarly, decreasing the distance to the nearest four-year college causes individuals to shift from not attending college to four-year college, and to shift from attending two-year college to attending four-year college. It does not cause any individuals to stop attending four-year college, or to shift between not attending college and two-year college.

These restrictions are motivated by individuals making their utility-maximizing choice of college attendance, with the utility from attending a college decreasing in the distance to the college.

Through the lens of the random utility model, Assumption UPM imposes restrictions on how instruments (e.g., distances to the nearest two-year college and four-year college) can affect individuals' utilities – the decrease of z_k to z'_k weakly increases the utility of k relative to all other choices, but does not affect the relative utility of other choices.

Assumption TRUM (Targeted Random Utility Model). *For all $i \in \mathcal{I}$, treatment status $D_i(z)$ satisfies*

$$D_i(z) = \arg\max_{d \in \mathcal{K}} V_{id}(z) \tag{1}$$

for all $z \in \mathcal{Z}$. Further, $V_{i0}(z) = 0$, and $V_{ik}(z) = U_{ik} - \mu_{ik}(z_k)$, where μ_{ik} is an increasing function of z_k, for all $k \in \mathcal{K} \setminus \{0\}$.

Proposition 1. *Assumptions UPM and TRUM are equivalent.*

Proof. I establish equivalence in two steps. First, I show that Assumption TRUM implies Assumption UPM. Second, I show that Assumption UPM implies Assumption TRUM; this second step is more involved, and I sketch the argument below with details in Appendix A.1.

Assumption TRUM $\Rightarrow$ Assumption UPM Take any $(z_k, z_{-k}), (z'_k, z_{-k}) \in \mathcal{Z}$ with $z'_k < z_k$. By Assumption TRUM, for all $i \in \mathcal{I}$, $V_{ik}(z'_k, z_{-k}) \geq V_{ik}(z_k, z_{-k})$, and for all $\ell \in \mathcal{K} \setminus \{k\}$, $V_{i\ell}(z'_k, z_{-k}) = V_{i\ell}(z_k, z_{-k})$. Applying Equation 1, for all $i \in \mathcal{I}$, $D_{ik}(z'_k, z_{-k}) \geq D_{ik}(z_k, z_{-k})$, and $D_{i\ell}(z'_k, z_{-k}) \leq D_{i\ell}(z_k, z_{-k})$ for all $\ell \in \mathcal{K} \setminus \{k\}$. Assumption UPM therefore holds.

Assumption TRUM $\Leftarrow$ Assumption UPM I proceed constructively, applying Assumption UPM to construct utility indices which I show satisfy Assumption TRUM. I do so in 3 steps:

First, I construct an individual's "willingness-to-pay" for each treatment k as the maximum value of z_k at which the individual would choose treatment k. By construction, when the instrument is elementwise greater than an individual's willingness-to-pay vector, the individual will choose 0. Applying Assumption UPM implies that this is if-and-only-if: willingness-to-pay therefore fully characterizes an individual's choice of 0 over all other treatments as a function of "prices".

Second, I construct an individual's "equivalent variation" for each treatment k, in units of the "price" of treatment 1, z_1, for the option to choose treatment k over treatment 1 (as if there was no option to choose treatment 0). I define this "equivalent variation" as the difference between the maximum "price" z_1 at which the individual chooses treatment 1, as a function of z_k, and the actual price of treatment 1, z_1. Applying Assumption UPM, I show that an individual chooses treatment 1 if and only if their willingness-to-pay for treatment 1 is positive, and their equivalent variation from all other treatments is negative.

Third, once again applying Assumption UPM, I show that an individual chooses treatment k if and only if their willingness-to-pay for treatment k is positive, their equivalent variation from treatment k is positive, and their equivalent variation from treatment k is higher than their equivalent variation from all other treatments j.

These three results immediately suggest a construction of utility indices satisfying Assumption TRUM. The utility index for treatment 0 is 0. The utility index for treatment 1 is the difference between willingness-to-pay for treatment 1 and the price of treatment 1, that is the compensating variation for choosing treatment 1 over treatment 0. The utility index for treatment k is the equivalent variation from treatment k minus the compensating variation from treatment 1, that is the difference between the individual's maximum "price" at which they would choose treatment 1 over treatment k (as a function of the price of treatment k) and the individual's willigness-to-pay for treatment 1. $\square$

Assumption TRUM is equivalent to assuming agents make choices as if they maximize utility and instruments are prices, as in the random utility model (McFadden, 1981). The equivalence between Assumption TRUM and UPM therefore provides a concrete interpretation of Assumption UPM.

1.2.2 Identical compliers, no defiers, and additive random utility

Conditional on Assumption UPM, which restricts the sign of treatment responses to changes in the value of the instrument, I consider three assumptions that restrict the heterogeneity of treatment responses.

Assumption IC (Identical Compliers)**.** *For all $z \in \mathcal{Z}$, all $k, \ell \in \mathcal{K} \setminus \{0\}$ where $k \neq \ell$, and for all bounded continuous $y(v) : B_+(\mathcal{Z}, \mathbb{R}^K) \to \mathbb{R}$, where $B_+(\mathcal{Z}, \mathbb{R}^K)$ is the set of increasing functions from $\mathcal{Z}$ to $\mathbb{R}^K$,*

$$\lim_{z'_k \uparrow z_k} \mathbf{E}[y(V_i) | D_i(z'_k, z_{-k}) = k, D_i(z_k, z_{-k}) = \ell] = \lim_{z'_\ell \downarrow z_\ell} \mathbf{E}[y(V_i) | D_i(z'_\ell, z_{-\ell}) = k, D_i(z_\ell, z_{-\ell}) = \ell]$$

One interpretation of Assumption IC is that it imposes "marginal monotonicity": interpreting $y(V_i)$ as a projection of an individual's "type" $V_i : \mathcal{Z} \to \mathbb{R}^K$, it imposes that the same average (projected) type $y(V_i)$ of individuals are marginal between treatment statuses k and ℓ, whether revealed by small changes in z_k or z_ℓ.

Assumption IC is closely related to comparable compliers from Mountjoy (2022); comparable compliers imposes the weaker assumption that Assumption IC holds only for $y(v) = \mathbf{E}[Y_{ik}|V_i = v]$, rather than for all bounded continuous $y(v)$.[2] Comparable compliers is a strictly weaker assumption, as, for example, it is satisfied whenever selection is independent of potential outcomes. I focus on identical compliers to separate assumptions on selection from assumptions on potential outcomes conditional on V_i (or, equivalently, $D_i(\cdot)$).

Assumption ND (No Defiers)**.** *For all $z, z' \in \mathcal{Z}$, $k, \ell \in \mathcal{K}$ with $k \neq \ell$*

$$\mathbf{P}\left[D_i(z) = k, D_i(z') = \ell\right] = 0 \text{ or } \mathbf{P}\left[D_i(z) = \ell, D_i(z') = k\right] = 0$$

Assumption ND imposes "global monotonicity": for any change in the value of the instruments, no two individuals have opposing treatment responses. Navjeevan & Pinto (2022) discuss identification under Assumption ND with discrete instruments.

Assumption ARUM (Additive Random Utility Model)**.** *For all $i \in \mathcal{I}$ and $z \in \mathcal{Z}$, treatment status satisfies Equation 1 where $V_{i0}(z) = 0$, and $V_{ik}(z) = U_{ik} - \mu_k(z_k)$ for all $k \in \mathcal{K} \setminus \{0\}$.*

Assumption ARUM imposes strong restrictions on heterogeneity of treatment responses: for any change in the value of the instruments, all individuals agree on the changes in relative utility across treatments. Lee & Salanié (2018) discuss identification of treatment effects under Assumption ARUM.

Assumptions IC, ND, and ARUM are redundant conditional on Assumption UPM for a binary treatment and instrument. In this case, Assumption UPM corresponds to the monotonicity assumption in Imbens & Angrist (1994), and increases in the binary instrument must shift agents away from control and into treatment. Without multivalued treatment, Assumption IC holds vacuously, Assumption ND is implied by monotonicity, and Vytlacil (2002) showed that Assumption ARUM is implied by monotonicity. However, with multiple treatments and multiple instruments, each of these assumptions puts additional restrictions on choices conditional on Assumption UPM.

[2]Substituting $y(v) = \mathbb{E}[Y_{ik}|V_i = v]$ into Assumption IC, and applying the law of iterated expectations, yields $\lim_{z'_k \uparrow z_k} \mathbf{E}\left[Y_{ik}|\, D_i(z'_k, z_\ell) = k, D_i(z_k, z_\ell) = \ell\right] = \lim_{z'_\ell \downarrow z_\ell} \mathbf{E}\left[Y_{ik}|\, D_i(z_k, z'_\ell) = k, D_i(z_k, z_\ell) = \ell\right]$, which is the comparable compliers assumption.

1.3 Relationships across assumptions on selection

Assumption ND can be equivalently stated in terms of the permissible directed graphs of flows between treatments in response to a change in the value of the instrument. Define the "treatment response graph" $G^{(z,z')}$ for a change in the value of the instrument from z to z', with element $(k,\ell) \in \mathcal{K}^2$ equal to $G^{(z,z')}_{(k,\ell)}$, by

$$G^{(z,z')}_{(k,\ell)} \equiv \mathbf{1}\{k \neq \ell \wedge \mathbf{P}[D_i(z) = k, D_i(z') = \ell] > 0\} \tag{2}$$

The element (k,ℓ) of the treatment response graph $G^{(z,z')}$ is 1 if the change of the instrument $z \to z'$ induces individuals to shift $k \to \ell$, and 0 otherwise.[3]

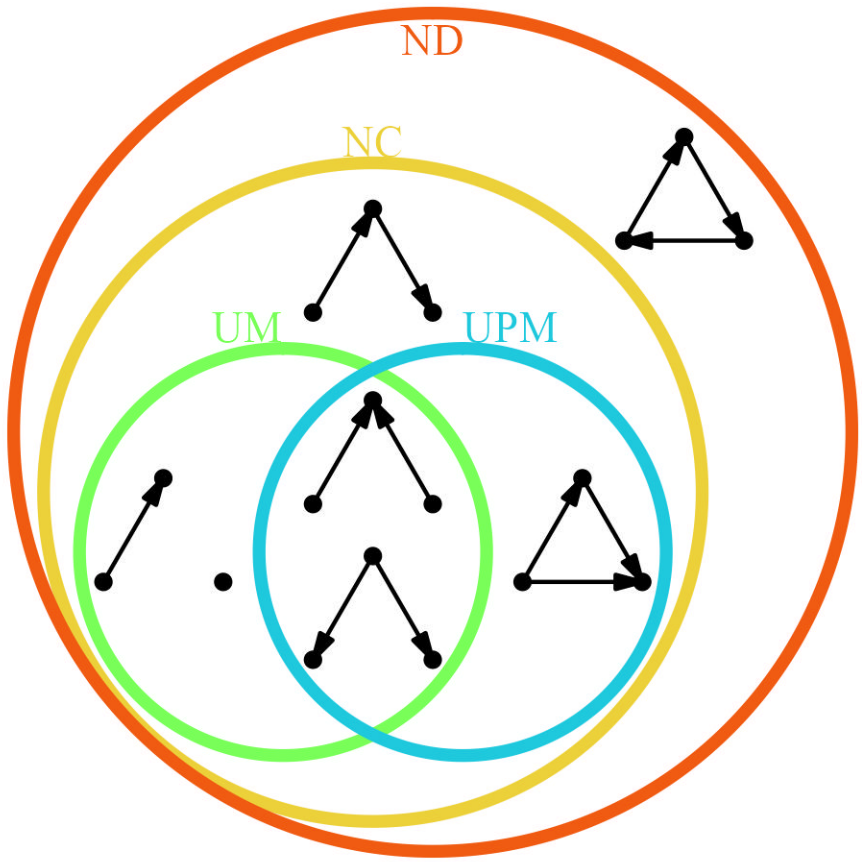
Assumption ND can be equivalently stated using the treatment response graph as follows: for any $k, \ell \in \mathcal{K}$, either $G^{(z,z')}_{(k,\ell)} = 0$ or $G^{(z,z')}_{(\ell,k)} = 0$. That is, there is no change in the value of the instrument from z to z' that both induces individuals to shift $d \to d'$ and also $d' \to d$.

Figure 1 plots the 6 unique treatment response graphs (up to permutations of nodes) that are consistent with no defiers when $K = 2$. Conditional on Assumption ND, additional assumptions place additional restrictions on the set of permissible treatment response graphs. I discuss the restrictions placed by 3 additional assumptions: no cycles, unordered monotonicity (Heckman & Pinto, 2018), and Assumption UPM.

First, "no cycles" ("NC") imposes that there are no cycles in the treatment response graph $G^{(z,z')}$. This is a stronger assumption than Assumption ND – it rules out the case, for example, where a change in the value of the instruments induces the treatment flows $\{0 \to a, a \to b, b \to 0\}$. Interpreting flows of individuals across treatments as a revealed preference, a cycle implies a form of preference heterogeneity: the change in the value of the instruments must have strictly increased the relative value of a to 0 for individuals shifting $0 \to a$, of b to a for agents shifting $a \to b$, and of 0 to b for agents shifting $b \to 0$. These changes in relative values are not simultaneously possible for a single agent. Cycles challenge identification of treatment

[3]The admissible treatment response graphs discussed here are closely related to admissible binary response matrices under unordered monotonicity in Heckman & Pinto (2018) and under minimal monotonicity (equivalent to no defiers) in Navjeevan & Pinto (2022); binary response matrices carry additional information on restrictions to responses across multiple values of an instrument, which Heckman & Pinto (2018) and Navjeevan & Pinto (2022) use to establish results on identification with discrete variation in instruments.

Figure 1: Feasible treatment response graphs conditional on no defiers



Notes: The set of unique feasible treatment response graphs $G^{(z,z')}$ for $K=2$, defined in Equation 2, up to permutations of nodes possible under Assumption ND are presented in this figure. The subset of these graphs that are feasible when either no cycles, unordered monotonicity, or Assumption UPM is also imposed are included in the associated labeled circle of the assumption.

effects in a manner similar to defiance: a cycle implies that unobserved flows of individuals across treatments are possible without any corresponding change in observed treatment probabilities.

Second, "unordered monotonicity" ("UM"), analyzed by Heckman & Pinto (2018), imposes that there is no treatment that experiences both individuals shifting into that treatment and individuals shifting out of that treatment in response to a change in the value of the instruments.[4] That is, for all $k \in \mathcal{K}$, either $\sum_{\ell\in\mathcal{K}} G^{(z,z')}_{(k,\ell)} = 0$

[4]Other work has analyzed assumptions that imply unordered monotonicity (Behaghel et al., 2013; Bhuller & Sigstad, 2023), and are therefore distinct from unordered partial monotonicity.

or $\sum_{\ell \in \mathcal{K}} G_{(\ell,k)}^{(z,z')} = 0$. UM strengthens no cycles, by effectively imposing that there are two tiers of treatments for any change in the instruments: those that receive flows of individuals from other treatments, and those that send flows of individuals to other treatments. It therefore rules out two sets of treatment response graphs consistent with no cycles which imply three tiers of treatments: $\{0 \to a, a \to b\}$, and $\{0 \to a, a \to b, 0 \to b\}$.

Third, Assumption UPM, analyzed by Mountjoy (2022), imposes that each instrument induces positive flows into its associated treatment and out of other treatments, but not between other treatments.[5] Assumption UPM implies, but is not implied by, additional restrictions on the set of permissible treatment response graphs. Consider as an example reducing both z_a and z_b. Reducing z_a must induce agents to shift $0 \to a$ and $b \to a$, while reducing z_b must induce agents to shift $a \to b$ and $0 \to b$; reducing z_a and then z_b must shift agents from $0 \to b$, while reducing z_b and then z_a must shift agents from $0 \to a$. When combined with no defiers, only three sets of flows are therefore possible from reducing both z_a and z_b: $\{0 \to a, 0 \to b\}$, $\{0 \to a, 0 \to b, a \to b\}$, and $\{0 \to a, 0 \to b, b \to a\}$. Applying the above reasoning to each possible change in the value of the instrument, Assumption ND and Assumption UPM jointly strengthen no cycles by implying transitivity of treatment flows: if a change in the value of the instruments shifts individuals $a \to b$ (i.e., strict revealed preference for b over a), and not from $a \to 0$ (i.e., weak revealed preference for a over 0), then it must also shift individuals from $0 \to b$ (i.e., strict revealed preference for b over 0).

That Assumptions UPM and ND jointly imply transitivity of treatment flows suggests an equivalence with Assumption ARUM: both transitivity and the "increasing differences" property of the additive random utility model (Lee & Salanié, 2023) are equivalent to the statement that any change in the value of the instrument $z \to z'$ induces a weak ordering of treatments through the associated treatment response graph $G^{(z,z')}$.

[5] While Assumption UPM only imposes that each instrument induces *weakly* positive flows into its associated treatment and out of other treatments, the version of Assumption UPM analyzed by Mountjoy (2022), or Assumption UPM coupled with Assumptions TRUM.1, TRUM.2, and TRUM.3, imposes *strictly* positive flows. In the analysis in this section, I implicitly assume strictly positive flows.

1.4 Technical assumptions on selection

I consider the following additional technical assumptions on the targeted random utility model in Assumption TRUM.

Assumption TRUM.1. *Let* $\mu_i(z) \equiv (\mu_{ik}(z_k))_{k=1}^K$*, such that* $\mu_i : \mathcal{Z} \to \mathbb{R}^K$*.* μ_i *has associated probability measure* F*, and* $\mu_i(z)$ *is continuously differentiable with respect to* z *for all* $i \in \mathcal{I}$ *with bounded derivative.*

Assumption TRUM.2. $U_i \equiv (U_{ik})_{k=1}^K$ *has strictly positive, bounded, and continuous density on* $\mathbb{R}^K$ *conditional on* μ_i*,* $f(\cdot|\mu_i)$*.*

Assumption TRUM.3. $\mu'_{ik}(z_k) > 0$ *for all* $i \in \mathcal{I}$*,* $z \in \mathcal{Z}$*,* $k \in \mathcal{K} \setminus \{0\}$*.*

I impose three additional technical restrictions on the targeted random utility model. Assumption TRUM.1 imposes that utility is continuously differentiable in the instrument. Assumption TRUM.2 ensures that for each "sensitivity" to the instrument μ_i and each value of the instrument z, there are positive densities of individuals U_i who are indifferent between any tuple of treatment statuses. Assumption TRUM.3 ensures that all individuals are responsive to both instruments.

1.5 Independence and assumptions on outcomes

Let Y_{id} be the outcome of individual i under treatment status d, such that the observed outcome $Y_i(Z_i) \equiv \sum_{d\in\mathcal{K}} D_{id}(Z_i)Y_{id}$. The econometrician observes $(Y_i(Z_i), D_i(Z_i), Z_i)$ for each individual $i \in \mathcal{I}$.

I make the following independence assumption.

Assumption I (Independence and Exclusion)**.** $Z_i \perp ((Y_{id})_{d=0}^K, V_i)$

Assumption I implies that the assigned value of the instrument Z_i is independent of potential outcomes and selection into treatment.

I apply throughout this paper two results that follow from Assumption I. First, $\mathbf{E}[D_{id}(Z_i)|Z_i = z] = \mathbf{E}[D_{id}(z)|Z_i = z] = \mathbf{E}[D_{id}(z)]$ for all $d \in \mathcal{K}$; I denote $P_d(z) \equiv \mathbf{E}[D_{id}(z)]$. Second, $\mathbf{E}[Y_i(Z_i)D_{id}(Z_i)|Z_i = z] = \mathbf{E}[Y_{id}D_{id}(z)|Z_i = z] = \mathbf{E}[Y_{id}D_{id}(z)]$ for all $d \in \mathcal{K}$; I denote $PY_d(z) \equiv \mathbf{E}[Y_{id}D_{id}(z)]$. By the above, $P_d(z)$ and $PY_d(z)$ are identified for all $z \in \mathcal{Z}$.

Technical assumptions on outcomes The analysis of identification in Sections 3 and 4 will require additional restrictions on the joint distribution of potential outcomes and latent utilities. I will maintain Assumption ARUM in Sections 3 and 4; I therefore omit the conditioning on the utility index in the density of U_i, $f(u)$.

I restrict expected potential outcomes conditional on $U_i = u$ as follows.

Assumption Y.1. *For all $d \in \mathcal{K}$, $\mathbf{E}[Y_{id}|U_i = u]$ is continuously differentiable with respect to u.*

Assumption Y.2. *Bounds $Y_{id} \in [\underline{Y}, \overline{Y}]$ are satisfied for all $d \in \mathcal{K}$, $i \in \mathcal{I}$.*

Lee & Salanié (2018) apply a weaker version of Assumption Y.1, assuming equicontinuity of $\mathbf{E}[Y_{id}|U_i = u]$ and $f(u)$, that avoids pathological cases by ensuring that $\mathbf{E}[Y_{id}|U_i = u]f(u)$ and $f(u)$ converge to limiting approximations at similar rates.

I apply the assumption that potential outcomes are bounded in Assumption Y.2 to derive bounds on LATE-2 in Section 4.

2 Equivalence across assumptions on selection

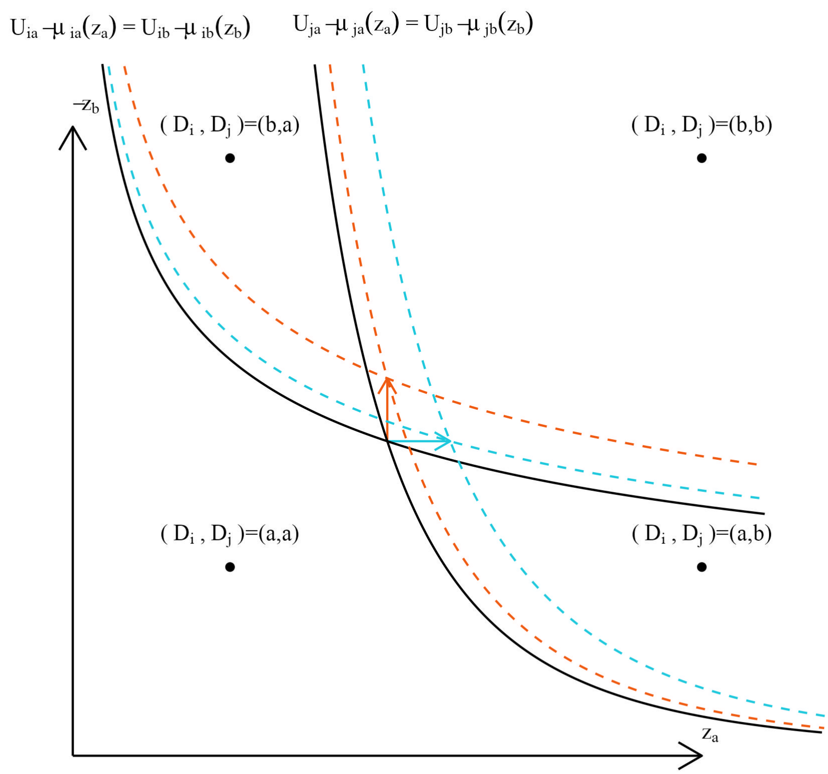
In this section, I show that Assumptions IC, ND, and ARUM are equivalent conditional on Assumption TRUM (or, equivalently, Assumption UPM) and Assumptions TRUM.1, TRUM.2, and TRUM.3.

The conditional equivalence of Assumptions IC, ND, and ARUM immediately implies that any results derived under one of these assumptions immediately holds under the others. For instance, testable restrictions of Assumption IC (Mountjoy, 2022), Assumption ND (Heckman & Vytlacil, 2005; Kitagawa, 2015; Rose & Shem-Tov, 2023), and Assumption ARUM (Allen & Rehbeck, 2019; Bhattacharya, 2023) are therefore testable restrictions of all three assumptions. This intuition applies to identification results for the marginal treatment effect (Section 3) and a local average treatment effect for two treatments (Section 4): while I establish identification under Assumptions TRUM and ARUM, by equivalence results hold under Assumption IC or Assumption ND instead of Assumption ARUM.

Proposition 2. *Suppose Assumptions TRUM, TRUM.1, TRUM.2, and TRUM.3 hold. Then Assumptions IC, ND, and ARUM are equivalent.*

Proof. To begin, I note that Assumption ARUM implies Assumption IC (as shown by Mountjoy, 2022) and Assumption ND (as shown by Lee & Salanié, 2023). Below, I show that either Assumption IC or Assumption ND implies homogeneous instrument sensitivity, which I show implies Assumption ARUM. Therefore, Assumptions IC, ND, and ARUM are equivalent.

Figure 2: Identical compliers or no defiers implies homogeneous instrument sensitivity



Notes: Two example indifference sets, corresponding to the set of values z for which individuals i and j are indifferent between treatments a and b, are plotted in this graph, which is centered at their point of intersection. Values of the treatment status of i and j as a function of z, (D_i, D_j), which contradict Assumption ND are plotted. Indifference sets through alternative values of z – with an increase in z_a and with a decrease in z_b – are plotted for both individuals of type-μ_i and type-μ_j, which contradict Assumption IC.

The intuition underlying the result that either Assumption IC or Assumption ND is equivalent to homogeneous instrument sensitivity, and therefore Assumption ARUM, is presented in Figure 2 for $K = 2$. Figure 2 builds closely on Mogstad et al.

(2021), applying their Figure 2 (and associated logic) to the setting with multivalued treatment and extending it from Assumption ND to Assumption IC.

First, Figure 2 presents an example with two defiers i and j for a given change in the instrument. Under Assumption TRUM, the implied sets of (z_a, z_b) for which defiers i and j are indifferent between treatment statuses must intersect, that is i and j have heterogeneous instrument sensitivity. One can similarly identify defiers from i-type and j-type individuals who differ in their instrument sensitivity.

Second, Figure 2 shows the effects of a small increase in z_a and a small decrease in z_b on individuals with instrument sensitivity characterized by (μ_{ia}, μ_{ib}) and (μ_{ja}, μ_{jb}); more of the μ_j-type individuals shift treatment status in response to the increase in z_a, while more of the μ_i-type individuals shift treatment status in response to the decrease in z_b.

Homogeneous instrument sensitivity $\Rightarrow$ Assumption ARUM I define homogeneous instrument sensitivity as

$$\frac{\mu'_{ik}(z_k)}{\mu'_{jk}(z_k)} = \frac{\mu'_{i\ell}(z_\ell)}{\mu'_{j\ell}(z_\ell)} \quad \forall i, j \in \mathcal{I}, k, \ell \in \mathcal{K} \setminus \{0\}, z \in \mathcal{Z}$$

These ratios are well defined under Assumption TRUM.3. As $\mathcal{Z}$ is a K-dimensional interval, if $(z'_k, z'_\ell, z_{-\{k,\ell\}}) \in \mathcal{Z}$ then $(z_k, z'_\ell, z_{-\{k,\ell\}}), (z'_k, z_\ell, z_{-\{k,\ell\}}) \in \mathcal{Z}$, and the above equality implies

$$\frac{\mu'_{ik}(z_k)}{\mu'_{jk}(z_k)} = \frac{\mu'_{ik}(z'_k)}{\mu'_{jk}(z'_k)} = \frac{\mu'_{i\ell}(z_\ell)}{\mu'_{j\ell}(z_\ell)} = \frac{\mu'_{i\ell}(z'_\ell)}{\mu'_{j\ell}(z'_\ell)} \quad \forall i, j \in \mathcal{I}, (z_k, z_\ell, z_{-\{k,\ell\}}), (z'_k, z'_\ell, z_{-\{k,\ell\}}) \in \mathcal{Z}$$

Fix any $j \in \mathcal{I}$. By continuous differentiability of $\mu_{ik}(z_k)$ by Assumption TRUM.1, rescaling $V_{ik}(z)$ implied by Assumption UPM in Proposition 1 by the i-specific constant $\frac{\mu'_{jk}(z_k)}{\mu'_{ik}(z_k)}$ yields the additive random utility model in Assumption ARUM with $\mu_k(z_k) \equiv \mu_{jk}(z_k)$.

Assumption ND $\Rightarrow$ Homogeneous instrument sensitivity By Assumptions TRUM.1, TRUM.2, and TRUM.3, for each type μ_i there are a positive density of marginal individuals at any value of the instrument z. Assumption TRUM.1 holds, and Proposition 2 of Mogstad et al. (2021) therefore implies that

$$\mu'_{ik}(z_k)\mu'_{j\ell}(z_\ell) = \mu'_{jk}(z_k)\mu'_{i\ell}(z_\ell)$$

For all $i, j \in \mathcal{I}$, $k, \ell \in \mathcal{K} \setminus \{0\}$, $z \in \mathcal{Z}$. By Assumption TRUM.2, this is equivalent to homogeneous instrument sensitivity.

Assumption IC $\Rightarrow$ Homogeneous instrument sensitivity The proof that Assumption IC implies homogeneous instrument sensitivity closely follows the logic in Figure 2, but the associated notation is cumbersome and the proof is left to Appendix A.2. □

3 Identification of the marginal treatment effect

In this section, I establish identification of the marginal treatment response ("MTR", Mogstad et al., 2018), and weighted averages of MTR. The MTR to d is the mean potential outcome under d among individuals with latent utility u.

$$\text{MTR}_d(u) \equiv \mathbf{E}[Y_{id}|U_i = u]$$

In particular, for any set of multiple treatments $\mathcal{K}_0 \subseteq \mathcal{K}$, and any $d \in \mathcal{K}_0$, I establish identification of average $\text{MTR}_d(U_i)$ across individuals indifferent among treatments in $\mathcal{K}_0$; under the additive random utility model in Assumption ARUM, such indifference is well defined. The difference between average marginal treatment responses under d and under another $\ell \in \mathcal{K}_0$ returns a weighted average of marginal treatment effects – it is the average treatment effect of d relative to ℓ among individuals indifferent among $\mathcal{K}_0$.

This result is a modest extension of existing identification results. Under Assumption UPM and a weaker version of Assumption IC, Mountjoy (2022) established identification when $|\mathcal{K}_0| = 2$.[6] Under Assumption TRUM and ARUM, and when the utility index $\mu(z)$ is known, Theorem 3.1 of Lee & Salanié (2018) established identification when $\mathcal{K}_0 = \mathcal{K}$; however, Lee & Salanié (2018) note that there is not generic guidance for local identification of utility indices. Under Assumption TRUM

[6] Although their analysis focuses on the case where the number of treatments $K = 2$, their analysis trivially extends to general case where $K > 2$.

and ARUM, Allen & Rehbeck (2019) and Bhattacharya (2023) establish identification of $\mu(z)$. I contribute by combining these results to establish identification when $\mathcal{K}_0 = \mathcal{K}$ and $\mu(z)$ is unknown, and in addition by establishing identification in cases where $2 < |\mathcal{K}_0| < K + 1$; for instance, average treatment effects among individuals indifferent between any triple of treatments are identified.

Proposition MTR (Identification of the marginal treatment response)**.** *Suppose Assumptions TRUM, TRUM.1, TRUM.2, TRUM.3, ARUM, I, and Y.1 hold. Take any $\mathcal{K}_0 \subseteq \mathcal{K}$ where $|\mathcal{K}_0| \geq 2$. Let $\mathcal{U}(\mathcal{K}_0) \equiv \{u : u_\ell - \mu_\ell(z_\ell) \geq u_d - \mu_d(z_d)\ \forall\ \ell \in \mathcal{K}_0, d \in \mathcal{K}\}$ be the set of values of latent utilities U_i corresponding to indifference among treatments in $\mathcal{K}_0$. Then, the average marginal treatment response $\mathbf{E}[Y_{id}|U_i \in \mathcal{U}(\mathcal{K}_0)]$ is identified for all $d \in \mathcal{K}_0$.*

Proof. Derivations and general expressions are left to Appendix A.3. I focus here on the case where $k \neq 0$ and $\mathcal{K}_0 = \mathcal{K}$, that is identification of the average outcome under treatment k for individuals indifferent between control and all treatments, $\text{MTR}_k(\mu(z))$.

$$\frac{\mu_k'(z_k)}{\mu_\ell'(z_\ell)} = \frac{\frac{\partial}{\partial z_k} P_\ell(z)}{\frac{\partial}{\partial z_\ell} P_k(z)} \tag{3}$$

$$f(\mu(z)) = -\frac{\partial}{\sum_{\ell \in \mathcal{K}\setminus\{0\}} \mu_\ell'(z_\ell)\partial z_\ell} \frac{\partial^{K-1}}{\prod_{\ell \in \mathcal{K}\setminus\{0,k\}} \mu_\ell'(z_\ell)\partial z_\ell} P_k(z) \tag{4}$$

$$\text{MTR}_k(\mu(z)) = \frac{\frac{\partial}{\sum_{\ell \in \mathcal{K}\setminus\{0\}} \mu_\ell'(z_\ell)\partial z_\ell} \frac{\partial^{K-1}}{\prod_{\ell \in \mathcal{K}\setminus\{0,k\}} \partial z_\ell} PY_k(z)}{\frac{\partial}{\sum_{\ell \in \mathcal{K}\setminus\{0\}} \mu_\ell'(z_\ell)\partial z_\ell} \frac{\partial^{K-1}}{\prod_{\ell \in \mathcal{K}\setminus\{0,k\}} \partial z_\ell} P_k(z)} \tag{5}$$

Equation 3 is derived by Allen & Rehbeck (2019) and Bhattacharya (2023), and establishes identification of $\mu(z)$ up to location ($\mu(z^0) = 0$ for fixed $z^0 \in \mathcal{Z}$) and scale ($\mu_1'(z_1^0) = 1$) normalizations.

Equation 4 is a special case of the identification of the density of marginal individuals derived by Lee & Salanié (2018) under known utility index; combined with Equation 3, it establishes identification of the density with an unknown utility index by K-differences. Intuitively, marginal individuals can be isolated as those who shift into treatment k in response to increasing the utility index of any other treatment ℓ (pushing individuals from ℓ to k), or in response to decreasing the utility index of all treatments (pushing individuals from 0 to k). Importantly, Equation 4 applies for the choice of any treatment k – we can isolate marginal individuals as those pushed into any given treatment through an increase in the utility index of any other treatment or a decrease in the utility indices of all treatments.

Once marginal individuals pushed into a given treatment k can be isolated, their mean potential outcomes under that treatment k can be identified as the "mass" of outcome under k pushed into treatment k normalized by the density of marginal individuals; Equation 5 expresses this result. $\square$

I present in Figure 3 the graphical intuition underlying the local identification result for the marginal treatment response in Equation 5 of Proposition MTR, for the case when $K = 2$. Figure 3 follows the approach to visualizing local identification in Mountjoy (2022). In each panel, I plot the values of the utility index $\mu(z)$ under small changes in the utility indices associated with the cross-partial derivatives in Equation 5. In turn, each value of the utility index is associated with treatment choices $D_i(z)$, characterized by three sets of inequalities:

$$
\begin{aligned}
D_i(z) = 0 &\Leftrightarrow U_{ia} < \mu_a(z_a), U_{ib} < \mu_b(z_b) \\
D_i(z) = a &\Leftrightarrow U_{ia} > \mu_a(z_a), U_{ib} - U_{ia} < \mu_b(z_b) - \mu_a(z_a) \\
D_i(z) = b &\Leftrightarrow U_{ib} > \mu_b(z_b), U_{ib} - U_{ia} > \mu_b(z_b) - \mu_a(z_a)
\end{aligned}
$$

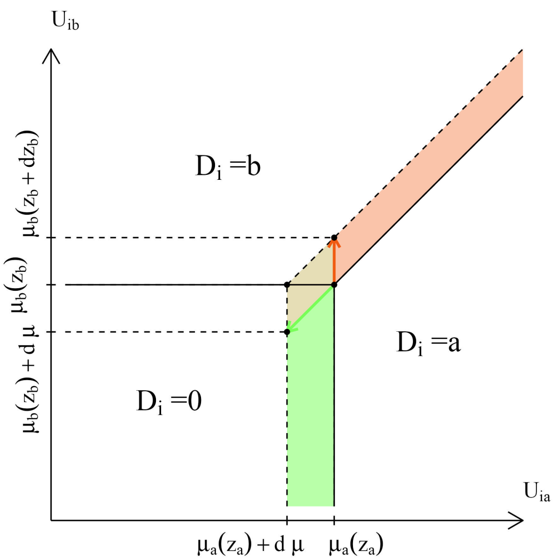
I color changes in the values of latent utilities U_i that shift into $D_i(z) = 0$, a, and b in Figures 3b, 3a, and 3c, respectively, with each shaded region then corresponding to a set of compliers for whom densities and mean potential outcomes are identified.

I briefly discuss Figure 3a, corresponding to identification of the mean potential outcome under treatment a ($\text{MTR}_a(\mu(z))$), to capture the underlying intuition. The ratio of local difference-in-differences which estimates $\text{MTR}_a(\mu(z))$ is constructed as follows. First, increasing z_b pushes individuals indifferent between treatments a and b into treatment a. Second, decreasing equally both $\mu_a(z_a)$ and $\mu_b(z_b)$ pushes individuals indifferent between treatments a and 0 into treatment a. Therefore, doing both of these additionally pushes individuals indifferent between a, b, and 0 into treatment a.

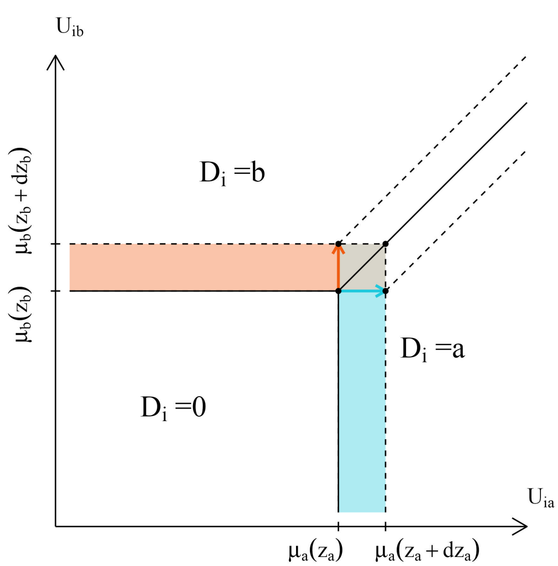
In many empirical applications, variation in instruments is discrete, rather than continuous. I therefore apply the intuition in Figure 3 to the case with discrete instruments in Section 4: with discrete instruments at the increments suggested by Figure 3, mean potential outcomes for selected groups of compliers is identified. However,

Figure 3: Local difference-in-differences identifies treatment effects for individuals indifferent across all treatment statuses

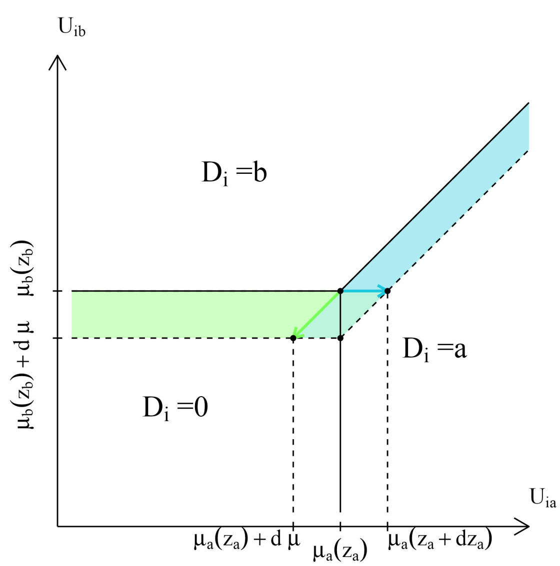
(a) Identification of $\mathbf{E}\left[Y_{ia}|U_i = \mu(z)\right]$



(b) Identification of $\mathbf{E}\left[Y_{i0}|U_i = \mu(z)\right]$



(c) Identification of $\mathbf{E}\left[Y_{ib}|U_i = \mu(z)\right]$



Notes: The set of individuals U_i with treatment status 0, a, and b under small changes in the value of the instruments z are plotted in each panel of this Figure. Shaded regions in panels (a), (b), and (c) correspond to changes in the set of individuals with treatment status a, 0, and b, respectively, under each of two possible changes in the values of the instruments or both.

these increments depend on the index μ, and ex-ante knowledge of this index is not, in general, plausible. I focus on price instruments in Section 4, in which case the absence of income effects implies $\mu(z) = z$.

4 Identification of a local average treatment effect for two treatments

In this section, I establish partial identification of a local average treatment effect for two treatments ("LATE-2") for a common set of compliers in a 3x3 factorial design cross randomizing zero, one, and two unit price increases for each treatment, when there are no income effects on treatment demand. In Section 4.1, I formalize the 3x3 design, and characterize all treatment response types $D_i(\cdot)$, including all eight complier groups. In Section 4.2, I derive bounds on LATE-2 for the union of two particular complier groups. I discuss the bounds and assumptions in Section 4.3.

4.1 Cross-randomized prices in the 3x3 experimental design

In addition to Assumptions TRUM and ARUM, I maintain the additional restriction on selection that the utility index μ is known. As μ is known, $\mu_k(z_k)$ can be used in place of z_k; I therefore assume without loss of generality that $\mu_k(z_k) = z_k$. When instruments are prices, this is equivalent to assuming there are no income effects, and I therefore interpret this assumption as such throughout this section.

Assumption NIE (No income effects)**.** *The utility index in Assumption ARUM, μ, is known; without loss of generality, $\mu(z) = z$.*

I restrict to the case where there are two treatments ($K = 2$), and the 3x3 factorial design cross randomizing zero, one, and two unit price increases for each treatment. After location and scale normalizations, this experimental design is equivalent to $\mathcal{Z} \equiv \{0, 1, 2\}^2$.

Assumption 3x3 (3x3 experimental design)**.** *The number of treatments $K = 2$, and $\mathcal{Z} \equiv \{0, 1, 2\}^2$.*

The choice of the 3x3 experimental design is motivated by Figure 3. Each panel of Figure 3 suggests that 4 points of support for $\mathcal{Z}$ are sufficient to identify the mean potential outcome among almost indifferent individuals. These 4 points of support correspond to a square (Figure 3b), vertically oriented parallelogram (Figure 3a), and horizontally oriented parallelogram (Figure 3c), for treatments 0, a, and b, respectively; the 3x3 unit grid in $\mathcal{Z}$ contains at least two of each of these sets of 4 points of support, which I show in Section 4.2 enables bounding a local average treatment effect for both treatments.

I will derive some results in the limit as the price changes in the 3x3 experimental design approach 0; for these results, I impose the following modified version of Assumption 3x3.

Assumption 3x3(ϵ). *Let $\epsilon > 0$. The number of treatments $K = 2$, and $\mathcal{Z}(\epsilon) \equiv \{0, \epsilon, 2\epsilon\}^2$.*

4.1.1 Treatment response types with no income effects

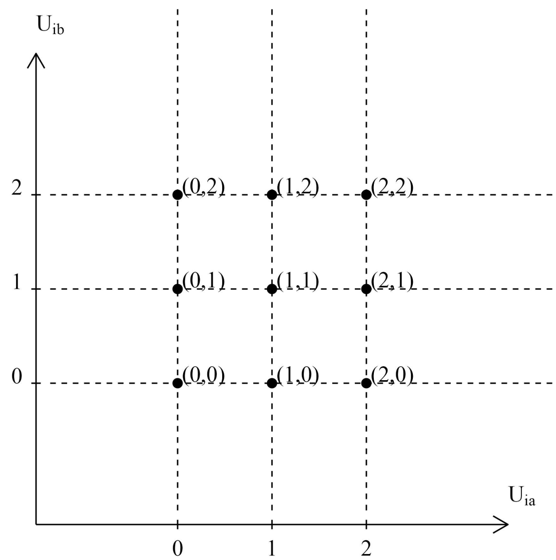
Assumptions NIE and 3x3 place substantial restrictions on the set of treatment response types $D_i(\cdot)$. Under Assumption NIE, a one unit increase in both z_a and z_b does not cause any individuals to switch from a to b or from b to a; note that absent Assumption NIE, one, but not both, of these switches may occur, while absent Assumption ARUM both of these switches may simultaneously occur. Many across-instrument comparisons in the 3x3 design involve one unit increases in both z_a and z_b, as the values of the instrument lie on the unit grid.

I characterize the full set of 19 treatment response types under Assumptions NIE and 3x3 in Figures 4a and 4b, following the approach to visualizing treatment response types in Lee & Salanié (2023). In Figure 4a, I plot the 9 instrument values in the 3x3 experimental design in the space of latent utilities U_i. I then plot the partitions of the space of U_i generated by the inequalities characterizing treatment choices $D_i(z)$ as a function of U_i for each $z \in \mathcal{Z}$ in Figure 4b; each resulting partition corresponds to a distinct treatment response type.

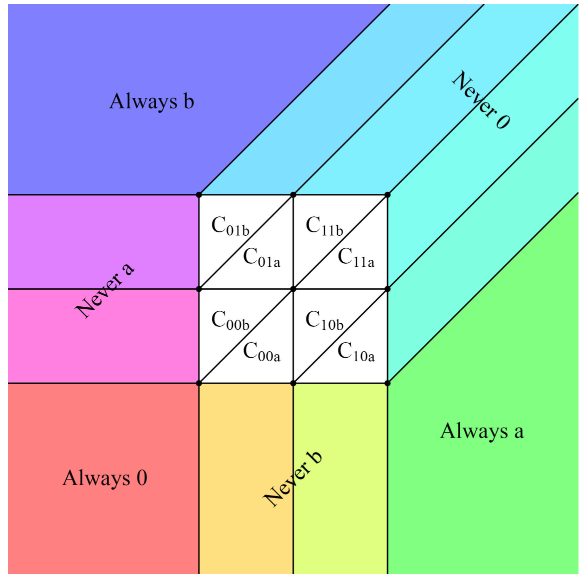
The 19 treatment response types are characterized as follows. First, individuals with sufficiently low latent utilities, sufficiently high latent utility for a, and sufficiently high latent utility for b are Always 0, Always a, and Always b, respectively. Individuals with sufficiently low latent utility for b, but intermediate latent utility for a, are Never b; Never b fall into two groups, those with latent utility for b between 0 and 1, and between 1 and 2. While Never a are characterized similarly, there are

Figure 4: Complier groups and their probabilities in the 3x3 design

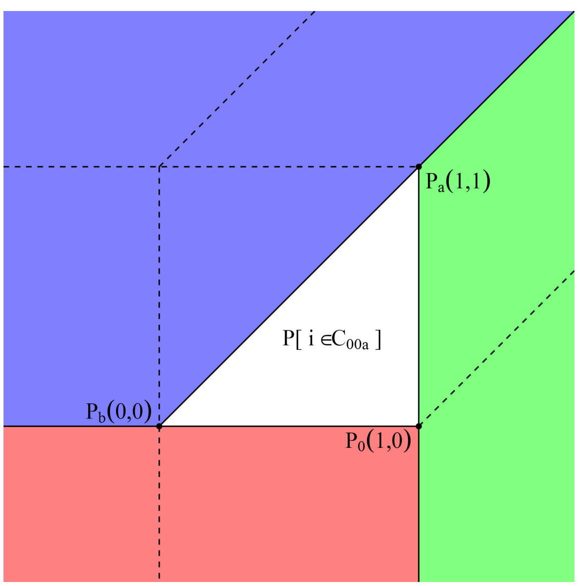
(a) 3x3 design ($\mathcal{Z} \equiv \{0, 1, 2\}^2$)



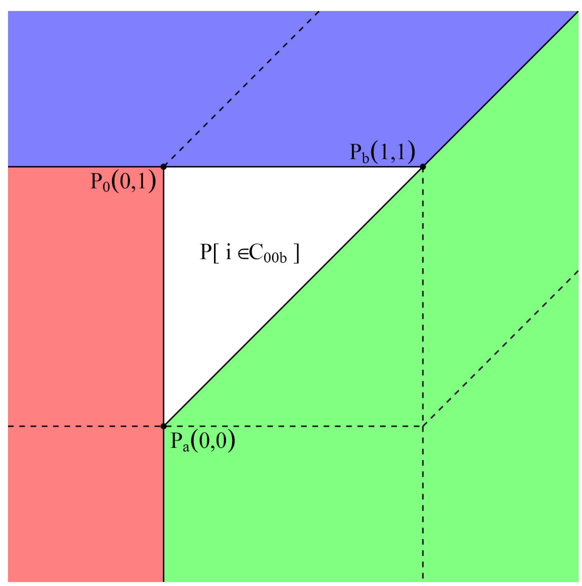
(b) Complier groups (C_{xyk})



(c) Identification of $\mathbf{P}\left[U_i \in C_{00a}\right]$



(d) Identification of $\mathbf{P}\left[U_i \in C_{00b}\right]$



Notes: The set of individuals U_i is plotted in each panel of this figure. Figure 4a plots the assigned values of the instrument in the 3x3 design in the space of individual latent utilities. Figure 4b plots the 19 treatment response types generated by the 3x3 design, including the 8 groups of compliers. Figures 4c and 4d demonstrate how the probabilities of each complier group are identified from observed treatment choice probabilities.

four groups of Never 0: those whose latent utility for a is between -2 and -1, -1 and 0, 0 and 1, and 1 and 2 less than their latent utility for b, corresponding to intervals between all possible values of $z_a - z_b$.

Lastly, and most importantly, there are 8 complier groups, who take up all three treatments at some value of the instruments. I index these complier groups by $(x, y, k) \in \{0,1\}^2 \times \{a, b\}$; each complier group C_{xyk} is uniquely characterized by

$$U_i \in C_{xyk} \Leftrightarrow \quad \begin{aligned} D_i(x,y) &= k \\ D_i(x+1,y) &= b \\ D_i(x,y+1) &= a \\ D_i(x+1,y+1) &= 0 \end{aligned}$$

Equivalently, $C_{xya} \equiv \{u : u_a \in [x, x+1], u_b \in [y, y+1], u_a - x \geq u_b - y\}$ and $C_{xyb} \equiv \{u : u_a \in [x, x+1], u_b \in [y, y+1], u_a - x \leq u_b - y\}$. Under Assumption 3x3($\epsilon$), I instead denote these complier groups $C_{xyk}(\epsilon)$.

These 8 complier groups are the only treatment response types that take up each treatment under observed values of the instruments, and therefore are the only treatment response types for which we can hope to bound all treatment effects, absent additional assumptions enabling extrapolation to unobservable counterfactuals.

4.2 LATE-2 theorem

Theorem LATE-2 (Informative partial identification of a local average treatment effect for two treatments with cross-randomized prices)**.** *Suppose prices of treatments a and b are randomized in the 3x3 experimental design (Assumption I, Section 1.5; Assumption 3x3, Section 4.1), individual choices satisfy the targeted additive random utility model with no income effects (Assumption TRUM, Section 1.2.1; Assumption ARUM, Section 1.2.2; Assumption NIE, Section 4.1), and additional technical assumptions on selection (Assumptions TRUM.1, TRUM.2, & TRUM.3, Section 1.4) and on outcomes (Assumption Y.1 & Y.2, Section 1.5) hold.*

Complier group probabilities are identified, and satisfy (Section 4.2.1):

$$\mathbf{P}[U_i \in C_{xya}] = 1 - P_b(x,y) - P_0(x+1,y) - P_a(x+1,y+1) \tag{6}$$

$$\mathbf{P}[U_i \in C_{xyb}] = 1 - P_a(x,y) - P_0(x,y+1) - P_b(x+1,y+1) \tag{7}$$

Mean potential outcomes for selected pairs of complier groups are identified, and satisfy (Section 4.2.2):

$$\mathbf{E}[Y_{i0}|U_i \in C_{xya} \cup C_{xyb}] = \frac{(PY_0(x+1,y+1) - PY_0(x+1,y)) - (PY_0(x,y+1) - PY_0(x,y))}{(P_0(x+1,y+1) - P_0(x+1,y)) - (P_0(x,y+1) - P_0(x,y))} \tag{8}$$

$$\mathbf{E}[Y_{ia}|U_i \in C_{xyb} \cup C_{x,y+1,a}] = \frac{(PY_a(x,y+1) - PY_a(x+1,y+2)) - (PY_a(x,y) - PY_a(x+1,y+1))}{(P_a(x,y+1) - P_a(x+1,y+2)) - (P_a(x,y) - P_a(x+1,y+1))} \tag{9}$$

$$\mathbf{E}[Y_{ib}|U_i \in C_{xya} \cup C_{x+1,y,b}] = \frac{(PY_b(x+1,y) - PY_b(x,y)) - (PY_b(x+2,y+1) - PY_b(x+1,y+1))}{(P_b(x+1,y) - P_b(x,y)) - (P_b(x+2,y+1) - P_b(x+1,y+1))} \tag{10}$$

Under monotonicity of local average treatment response (Assumption MLATR, Section 4.2.4), mean potential outcomes for $C_{01a} \cup C_{10b}$ under 0, a, and b satisfy the following bounds (Section 4.2.4); in the limit as the difference between randomized prices approaches 0, both bounds converge to the marginal treatment response, and monotonicity of local average treatment response holds to first order.

$$\begin{aligned}
&\mathbf{E}[Y_{i0}|U_i \in C_{01a} \cup C_{10b}] \in \\
&\Bigg[\min\Bigg\{\mathbf{E}[Y_{i0}|U_i \in C_{01a} \cup C_{01b}] - \left|\frac{\mathbf{P}[U_i \in C_{01a}]}{\mathbf{P}[U_i \in C_{01a} \cup C_{10b}]} - \frac{\mathbf{P}[U_i \in C_{01a}]}{\mathbf{P}[U_i \in C_{01a} \cup C_{01b}]}\right|(\overline{Y} - \underline{Y}), \\
&\quad \mathbf{E}[Y_{i0}|U_i \in C_{10a} \cup C_{10b}] - \left|\frac{\mathbf{P}[U_i \in C_{10b}]}{\mathbf{P}[U_i \in C_{01a} \cup C_{10b}]} - \frac{\mathbf{P}[U_i \in C_{10b}]}{\mathbf{P}[U_i \in C_{10a} \cup C_{10b}]}\right|(\overline{Y} - \underline{Y})\Bigg\}, \\
&\max\Bigg\{\mathbf{E}[Y_{i0}|U_i \in C_{01a} \cup C_{01b}] + \left|\frac{\mathbf{P}[U_i \in C_{01a}]}{\mathbf{P}[U_i \in C_{01a} \cup C_{10b}]} - \frac{\mathbf{P}[U_i \in C_{01a}]}{\mathbf{P}[U_i \in C_{01a} \cup C_{01b}]}\right|(\overline{Y} - \underline{Y}), \\
&\quad \mathbf{E}[Y_{i0}|U_i \in C_{10a} \cup C_{10b}] + \left|\frac{\mathbf{P}[U_i \in C_{10b}]}{\mathbf{P}[U_i \in C_{01a} \cup C_{10b}]} - \frac{\mathbf{P}[U_i \in C_{10b}]}{\mathbf{P}[U_i \in C_{10a} \cup C_{10b}]}\right|(\overline{Y} - \underline{Y})\Bigg\}\Bigg]
\end{aligned} \tag{11}$$

$$
\begin{aligned}
&\mathbf{E}[Y_{ia}|U_i \in C_{01a} \cup C_{10b}] \in \\
&\Bigg[\min\Bigg\{\mathbf{E}[Y_{ia}|U_i \in C_{00b} \cup C_{01a}] - \left|\frac{\mathbf{P}[U_i \in C_{01a}]}{\mathbf{P}[U_i \in C_{01a} \cup C_{10b}]} - \frac{\mathbf{P}[U_i \in C_{01a}]}{\mathbf{P}[U_i \in C_{00b} \cup C_{01a}]}\right| (\overline{Y} - \underline{Y}), \\
&\quad \mathbf{E}[Y_{ia}|U_i \in C_{10b} \cup C_{11a}] - \left|\frac{\mathbf{P}[U_i \in C_{10b}]}{\mathbf{P}[U_i \in C_{01a} \cup C_{10b}]} - \frac{\mathbf{P}[U_i \in C_{10b}]}{\mathbf{P}[U_i \in C_{10b} \cup C_{11a}]}\right| (\overline{Y} - \underline{Y})\Bigg\}, \\
&\max\Bigg\{\mathbf{E}[Y_{ia}|U_i \in C_{00b} \cup C_{01a}] + \left|\frac{\mathbf{P}[U_i \in C_{01a}]}{\mathbf{P}[U_i \in C_{01a} \cup C_{10b}]} - \frac{\mathbf{P}[U_i \in C_{01a}]}{\mathbf{P}[U_i \in C_{00b} \cup C_{01a}]}\right| (\overline{Y} - \underline{Y}), \\
&\quad \mathbf{E}[Y_{ia}|U_i \in C_{10b} \cup C_{11a}] + \left|\frac{\mathbf{P}[U_i \in C_{10b}]}{\mathbf{P}[U_i \in C_{01a} \cup C_{10b}]} - \frac{\mathbf{P}[U_i \in C_{10b}]}{\mathbf{P}[U_i \in C_{10b} \cup C_{11a}]}\right| (\overline{Y} - \underline{Y})\Bigg\}\Bigg]
\end{aligned} \tag{12}
$$

$$
\begin{aligned}
&\mathbf{E}[Y_{ib}|U_i \in C_{01a} \cup C_{10b}] \in \\
&\Bigg[\min\Bigg\{\mathbf{E}[Y_{ib}|U_i \in C_{01a} \cup C_{11b}] - \left|\frac{\mathbf{P}[U_i \in C_{01a}]}{\mathbf{P}[U_i \in C_{01a} \cup C_{10b}]} - \frac{\mathbf{P}[U_i \in C_{01a}]}{\mathbf{P}[U_i \in C_{01a} \cup C_{11b}]}\right| (\overline{Y} - \underline{Y}), \\
&\quad \mathbf{E}[Y_{ib}|U_i \in C_{00a} \cup C_{10b}] - \left|\frac{\mathbf{P}[U_i \in C_{10b}]}{\mathbf{P}[U_i \in C_{01a} \cup C_{10b}]} - \frac{\mathbf{P}[U_i \in C_{10b}]}{\mathbf{P}[U_i \in C_{00a} \cup C_{10b}]}\right| (\overline{Y} - \underline{Y})\Bigg\}, \\
&\max\Bigg\{\mathbf{E}[Y_{ib}|U_i \in C_{01a} \cup C_{11b}] + \left|\frac{\mathbf{P}[U_i \in C_{01a}]}{\mathbf{P}[U_i \in C_{01a} \cup C_{10b}]} - \frac{\mathbf{P}[U_i \in C_{01a}]}{\mathbf{P}[U_i \in C_{01a} \cup C_{11b}]}\right| (\overline{Y} - \underline{Y}), \\
&\quad \mathbf{E}[Y_{ib}|U_i \in C_{00a} \cup C_{10b}] + \left|\frac{\mathbf{P}[U_i \in C_{10b}]}{\mathbf{P}[U_i \in C_{01a} \cup C_{10b}]} - \frac{\mathbf{P}[U_i \in C_{10b}]}{\mathbf{P}[U_i \in C_{00a} \cup C_{10b}]}\right| (\overline{Y} - \underline{Y})\Bigg\}\Bigg]
\end{aligned} \tag{13}
$$

I prove Theorem LATE-2 in four steps. In Section 4.2.1, I establish identification of complier group probabilities. In Section 4.2.2, I establish identification of mean potential outcomes for selected pairs of complier groups. In Section 4.2.3, I justify the need for an additional assumption by establishing that absent monotonicity of local average treatment response ("MLATR"), bounds may be uninformative in the limit as the difference between randomized prices approaches 0. In Section 4.2.4, I derive bounds on LATE-2 under MLATR. I show these bounds converge to the marginal treatment response, and correspondingly that MLATR holds to first order, as the difference between randomized prices approaches 0.

Theorem LATE-2 establishes bounds on mean potential outcomes under 0, a, and b for $C_{01a} \cup C_{10b}$, implying bounds the local average treatment effect of a and b for $C_{01a} \cup C_{10b}$.

4.2.1 Complier group probabilities

I present a graphical proof of identification of complier group probabilities for C_{xya} (Equation 6) and C_{xyb} (Equation 7) in Figures 4c and 4d, respectively; I present an alternative proof in Appendix A.4.

The intuition underlying the identification argument in Figure 4c, for complier group C_{xya}, is as follows; the intuition parallels the two unique characterizations of complier groups in Section 4.1.1, one based on their treatment response type and the other based on their latent utilities. C_{xya} comprises the only individuals for whom $D_i(x, y) \neq b$ (instead, they choose a), $D_i(x + 1, y) \neq 0$ (instead, they choose b), and $D_i(x+1, y+1) \neq a$ (instead, they choose 0); this immediately yields Equation 6. This characterization can be interpreted as follows. The willingness to pay of individuals in C_{xya} for a is high enough that they choose it at (x, y), but low enough that increasing the price of a by one unit (to $x + 1$) pushes them to prefer both 0 and b. In addition, their willingness to pay for b is high enough that they choose it at $(x + 1, y)$, but low enough that increasing the price of b by one unit (to $y + 1$) pushes them to prefer 0.

To gain additional intuition, I add Equations 6 and 7 together, and apply the identity $P_0(z) + P_a(z) + P_b(z) = 1$, yielding

$$\mathbf{P}\left[U_i \in C_{xya} \cup C_{xyb}\right] = (P_0(x + 1, y + 1) - P_0(x + 1, y)) - (P_0(x, y + 1) - P_0(x, y)) \tag{14}$$

$$\mathbf{P}\left[U_i \in C_{xyb} \cup C_{x,y+1,a}\right] = (P_a(x, y + 1) - P_a(x + 1, y + 2)) - (P_a(x, y) - P_a(x + 1, y + 1)) \tag{15}$$

$$\mathbf{P}\left[U_i \in C_{xya} \cup C_{x+1,y,b}\right] = (P_b(x + 1, y) - P_b(x, y)) - (P_b(x + 2, y + 1) - P_b(x + 1, y + 1)) \tag{16}$$

These equations are difference-in-difference estimands, and intuitively are discrete versions of the "local difference-in-differences" expression for the density of U_i in Equation 4. That these formulas depend only on P_0, P_a, and P_b, respectively, suggests a related approach can be used to identify mean potential outcomes under 0, a, and b, respectively, for the complier groups suggested by Equations 14, 15, and 16; that is, a discrete version of Equation 5 and Figure 3. I implement this approach in Section 4.2.2, and apply it to bound a local average treatment effect in Sections 4.2.3 and 4.2.4.

4.2.2 Mean potential outcomes for selected complier groups

I establish identification of mean potential outcomes for selected complier groups in Equations 8, 9, and 10. While complier group probabilities can be derived from difference-in-differences of treatment choice indicators (Equations 14, 15, and 16), mean potential outcomes are derived from the ratio of difference-in-differences of treatment choice indicators times outcomes to difference-in-differences of treatment choice indicators. This is a natural extension of the approach of Abadie (2002), who similarly establish identification of average outcomes under control and treatment for compliers with one treatment.

Figures 5a, 5c, and 5e provide a graphical intuition underlying the identification results in Equations 8, 9, and 10, respectively; the approach is simply a discrete version of Figure 3, which presents a graphical argument for identification of the marginal treatment response. I present a formal derivation of Equation 9 in Appendix A.5; Equation 10 holds symmetrically, and Equation 8 follows from a similar derivation.

Mean potential outcomes under 0, a, and b are not identified for all 8 complier groups in the 3x3 design, and in fact are not point identified for any single one of the 8 complier groups. Figures 5b, 5d, and 5f show the 4, 2, and 2 pairs of complier groups for whom the mean potential outcome under 0, a, and b are, respectively, identified by Equations 8, 9, and 10. The intersection of these pairs, $C_{01a} \cup C_{10b}$, represents a natural group of compliers for whom to bound a local average treatment effect.

4.2.3 Uninformative bounds on LATE-2

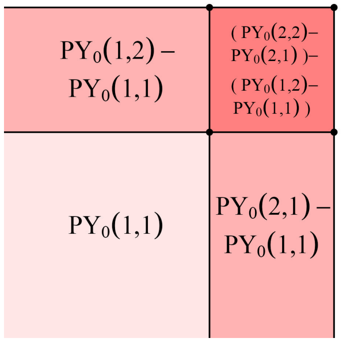
Absent additional assumptions, bounding mean potential outcomes under 0, a, or b for $C_{01a} \cup C_{10b}$, as suggested by Figure 5, appears challenging. Specifically, we are only able to estimate mean potential outcome under 0 for $C_{01a} \cup C_{01b}$ (rather than C_{01a}), and for $C_{10a} \cup C_{10b}$ (rather than C_{10b}). However, these averages could be rationalized by very low (or very high) mean potential outcomes for C_{01a} and C_{10b}, jointly with very high (or very low) mean potential outcomes for C_{01b} and C_{10a}.

Unfortunately, and consistent with this intuition, worst-case bounds on a local average treatment effect are completely uninformative in the limit as the price increments in the 3x3 experimental design approach 0.

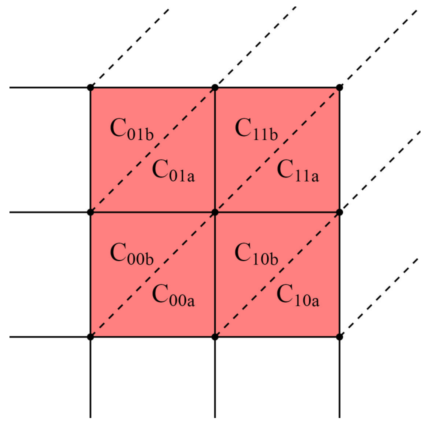
Proposition 3. *Suppose Assumptions I, 3x3(ϵ), TRUM, ARUM, NIE, TRUM.1, TRUM.2, TRUM.3, Y.1, and Y.2 hold. Suppose* $MTR_0(0) = \frac{1}{2}(\underline{Y} + \overline{Y})$. *Then the*

Figure 5: Set identification of a local average treatment effect in the 3x3 design

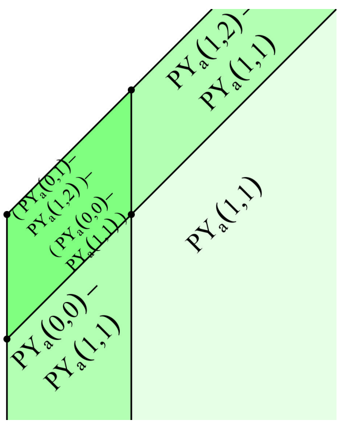
(a) $\mathbf{E}\left[Y_{i0}|U_i \in C_{11a} \cup C_{11b}\right]$



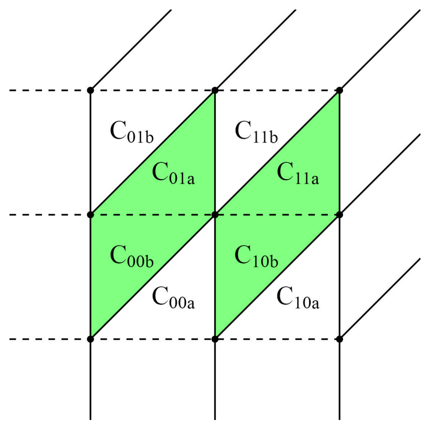
(b) Mean potential outcome under 0 identified for 4 complier groups



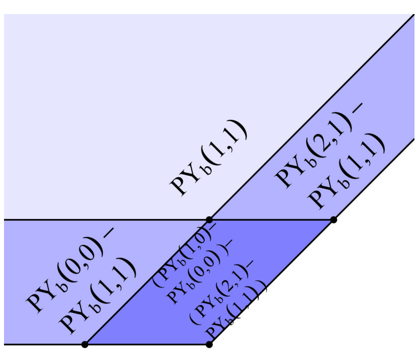
(c) $\mathbf{E}\left[Y_{ia}|U_i \in C_{00b} \cup C_{01a}\right]$



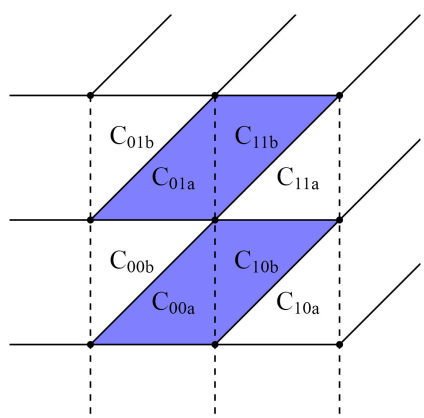
(d) Mean potential outcome under a identified for 2 complier groups



(e) $\mathbf{E}\left[Y_{ib}|U_i \in C_{00a} \cup C_{10b}\right]$



(f) Mean potential outcome under b identified for 2 complier groups



limit as $\epsilon \to 0$ of Manski (1990) bounds on $\mathbf{E}[Y_{i0}|U_i \in C_{01a}(\epsilon) \cup C_{10b}(\epsilon)]$ approaches $[\underline{Y}, \overline{Y}]$.

Proof. A formal proof is presented in Appendix A.6. Less formally, suppose complier group mean potential outcomes under 0 for $C_{01a}(\epsilon)$ and $C_{10b}(\epsilon)$ are equal to $\underline{Y}$, and for $C_{01b}(\epsilon)$ and $C_{10a}(\epsilon)$ are equal to $\overline{Y}$. Note that, in general, there is no guarantee that this would be consistent with the identified mean potential outcomes under 0 for $C_{01a}(\epsilon) \cup C_{01b}(\epsilon)$ and $C_{10a}(\epsilon) \cup C_{10b}(\epsilon)$. However, in the limit as $\epsilon \to 0$, the density of latent utilities approaches uniform and complier group probabilities approach $\frac{\epsilon^2}{2} f(0)$, while identified mean potential outcomes under 0 approach $\mathrm{MTR}_0(0)$. Identified complier group mean potential outcomes under 0 therefore converge to

$$\begin{aligned}\lim_{\epsilon \to 0} \mathbf{E}[Y_{i0}|U_i \in C_{xya} \cup C_{xyb}] &= \frac{1}{2} \lim_{\epsilon \to 0} \mathbf{E}[Y_{i0}|U_i \in C_{xya}] + \frac{1}{2} \lim_{\epsilon \to 0} \mathbf{E}[Y_{i0}|U_i \in C_{xyb}] \\ &= \mathrm{MTR}_0(0) = \frac{1}{2}(\underline{Y} + \overline{Y})\end{aligned}$$

This is consistent with the supposed complier group mean potential outcomes under 0, which realize the proposed lower bound of $\underline{Y}$. □

4.2.4 Informative bounds on LATE-2 under monotone local average treatment response

Absent additional assumptions, and focusing on the mean potential outcome under 0 for concreteness, the lower bound on mean potential outcomes for $C_{01b} \cup C_{10a}$ was uninformative as observed potential outcomes could be rationalized by mean potential outcomes in C_{01a} and C_{10b} being much lower than those in C_{01b} and C_{10a}. This possibility requires a particularly pernicious, and likely unrealistic, form of selection: an increase in willingness-to-pay for a by 1 unit and a decrease in willingness-to-pay for b by 1 unit is both associated with a large decrease in mean potential outcomes (going from C_{01a} to C_{10a}) and also a large increase in mean potential outcomes (going from C_{01b} to C_{10b}). Assuming away the possibility of such pernicious selection may yield tighter bounds.

I therefore propose bounds on LATE-2 under a multidimensional generalization of monotone treatment selection (Manski & Pepper, 2000), monotone local average treatment response ("MLATR"). MLATR rules out the possibility that a given complier group translation can increase potential outcomes under treatment d in one complier group but decrease potential outcomes under treatment d in another complier group. For comparison, monotone treatment selection imposes that the mean potential outcome under treatment d is increasing in scalar latent utility, and therefore implies MLATR.[7]

Assumption MLATR (Monotone local average treatment response)**.** *Let* $d \in \{0, a, b\}$*, then*

$$\begin{aligned}\mathbf{E}[Y_{id}|U_i \in C_{xyk}] &\geq \mathbf{E}[Y_{id}|U_i \in C_{x+w_a,y+w_b,k}] \\ &\Leftrightarrow \mathbf{E}[Y_{id}|U_i \in C_{x'y'k}] \geq \mathbf{E}[Y_{id}|U_i \in C_{x'+w_a,y'+w_b,k}]\end{aligned}$$

Proposition 4. *Suppose Assumptions I, 3x3(ϵ), TRUM, ARUM, NIE, TRUM.1, TRUM.2, TRUM.3, Y.1, and Y.2 hold. Then Assumption MLATR holds to first order in the limit as $\epsilon \to 0$.*

Proof. A formal proof is presented in Appendix A.7. Less formally, note that the marginal treatment response and density of latent utilities are continuously differentiable by assumption; as a result, as $\epsilon \to 0$, the inequalities in Assumption MLATR depend only on the sign of the directional derivative of marginal treatment response with respect to (u_a, u_b) along the vector (w_a, w_b), which approaches a common value. □

Proposition 5. *Suppose Assumptions I, 3x3, TRUM, ARUM, NIE, TRUM.1, TRUM.2, TRUM.3, Y.1, Y.2, and MLATR hold. Then the bounds in Equations 11, 12, and 13 are satisfied.*

Proof. I derive the lower bound in Equation 11; the upper bound, and in turn Equations 12 and 13, follow by a symmetric argument.

First, I decompose the mean potential outcome under 0 in $C_{01a} \cup C_{10b}$ into a weighted average of mean potential outcomes in C_{01a} and C_{10b}.

$$\begin{aligned}&\mathbf{E}[Y_{i0}|U_i \in C_{01a} \cup C_{10b}] \\ &\quad = \frac{\mathbf{P}[U_i \in C_{01a}]}{\mathbf{P}[U_i \in C_{01a} \cup C_{10b}]}\mathbf{E}[Y_{i0}|U_i \in C_{01a}] + \frac{\mathbf{P}[U_i \in C_{10b}]}{\mathbf{P}[U_i \in C_{01a} \cup C_{10b}]}\mathbf{E}[Y_{i0}|U_i \in C_{10b}]\end{aligned}$$

[7] With two-or-higher-dimensional latent utility U_i, which this paper concerns, it is no longer necessarily the case that monotone marginal treatment response with respect to elementwise comparison of latent utilities is sufficient for MLATR, unless the density of latent utility f is uniform.

Second, I note that either $\mathbf{E}[Y_{i0}|U_i \in C_{01b}] \leq \mathbf{E}[Y_{i0}|U_i \in C_{10b}]$ or $\mathbf{E}[Y_{i0}|U_i \in C_{01b}] \geq \mathbf{E}[Y_{i0}|U_i \in C_{10b}]$. If the latter, then by Assumption MLATR, $\mathbf{E}[Y_{i0}|U_i \in C_{01a}] \geq \mathbf{E}[Y_{i0}|U_i \in C_{10a}]$. As a consequence, either $\mathbf{E}[Y_{i0}|U_i \in C_{10b}] \geq \mathbf{E}[Y_{i0}|U_i \in C_{01b}]$ or $\mathbf{E}[Y_{i0}|U_i \in C_{01a}] \geq \mathbf{E}[Y_{i0}|U_i \in C_{10a}]$. I substitute each of these inequalities into the above equation, yielding

$$\begin{aligned}&\mathbf{E}[Y_{i0}|U_i \in C_{01a} \cup C_{10b}] \geq \\ &\min\left\{\frac{\mathbf{P}[U_i \in C_{01a}]}{\mathbf{P}[U_i \in C_{01a} \cup C_{10b}]}\mathbf{E}[Y_{i0}|U_i \in C_{01a}] + \frac{\mathbf{P}[U_i \in C_{10b}]}{\mathbf{P}[U_i \in C_{01a} \cup C_{10b}]}\mathbf{E}[Y_{i0}|U_i \in C_{01b}],\right. \\ &\qquad \left.\frac{\mathbf{P}[U_i \in C_{01a}]}{\mathbf{P}[U_i \in C_{01a} \cup C_{10b}]}\mathbf{E}[Y_{i0}|U_i \in C_{10a}] + \frac{\mathbf{P}[U_i \in C_{10b}]}{\mathbf{P}[U_i \in C_{01a} \cup C_{10b}]}\mathbf{E}[Y_{i0}|U_i \in C_{10b}]\right\}\end{aligned}$$

Lastly, I decompose the mean potential outcome under 0 for $C_{01a} \cup C_{01b}$ and $C_{10a} \cup C_{10b}$ into weighted averages of mean potential outcomes in C_{01a} and in C_{01b}, and in C_{10a} and in C_{10b}, respectively. Applying these decompositions, after adding and substracting the mean potential outcomes from the terms on the right hand side of the inequality, yields

$$\begin{aligned}&\mathbf{E}[Y_{i0}|U_i \in C_{01a} \cup C_{10b}] \geq \\ &\min\left\{\mathbf{E}[Y_{i0}|U_i \in C_{01a} \cup C_{01b}] + \left(\frac{\mathbf{P}[U_i \in C_{01a}]}{\mathbf{P}[U_i \in C_{01a} \cup C_{10b}]} - \frac{\mathbf{P}[U_i \in C_{01a}]}{\mathbf{P}[U_i \in C_{01a} \cup C_{01b}]}\right)(\mathbf{E}[Y_{i0}|U_i \in C_{01a}] - \mathbf{E}[Y_{i0}|U_i \in C_{01b}]),\right. \\ &\left.\mathbf{E}[Y_{i0}|U_i \in C_{10a} \cup C_{10b}] + \left(\frac{\mathbf{P}[U_i \in C_{10b}]}{\mathbf{P}[U_i \in C_{01a} \cup C_{10b}]} - \frac{\mathbf{P}[U_i \in C_{10b}]}{\mathbf{P}[U_i \in 10a \cup C_{10b}]}\right)(\mathbf{E}[Y_{i0}|U_i \in C_{10b}] - \mathbf{E}[Y_{i0}|U_i \in C_{10a}])\right\}\end{aligned}$$

Lastly, I substitute worst case bounds of $[\underline{Y} - \overline{Y}, \overline{Y} - \underline{Y}]$ for $\mathbf{E}[Y_{i0}|U_i \in C_{01a}] - \mathbf{E}[Y_{i0}|U_i \in C_{01b}]$ and $\mathbf{E}[Y_{i0}|U_i \in C_{10b}] - \mathbf{E}[Y_{i0}|U_i \in C_{10a}]$, which yields the lower bound in Equation 11. $\square$

Proposition 6. *Suppose Assumptions I, 3x3(ϵ), TRUM, ARUM, NIE, TRUM.1, TRUM.2, TRUM.3, Y.1, Y.2, and MLATR hold. Then the bounds in Equations 11, 12, and 13 approach $MTR_0(0)$, $MTR_a(0)$, and $MTR_b(0)$, respectively, as $\epsilon \to 0$.*

Proof. A formal proof is presented in Appendix A.8. Less formally, as $\epsilon \to 0$, the density of latent utilities approaches uniform and complier group probabilities therefore converge to $\frac{\epsilon^2}{2}f(0)$, while complier group mean potential outcomes under d converge to $\text{MTR}_d(0)$. Substituting these limits into the bounds in Equations 11, 12, and 13 yields the desired result. $\square$

The simultaneous weakness (Proposition 4) and strength (Proposition 6) of Assumption MLATR motivate the focus on bounds on LATE-2 derived under Assumption MLATR in Proposition 5. Weakness, in that Proposition 4 implies that Assumption MLATR holds for small randomized price changes. Strength, in that Proposition 6 implies that Assumption MLATR yields bounds on LATE-2 that converge to MTE for small randomized price changes, while absent Assumption MLATR there is no guarantee that bounds are even informative.

4.3 Discussion

This section establishes informative partial identification of LATE-2 under the 3x3 experimental design with no income effects on treatment demand; I make three key observations about this result. First, it imposes a strong assumption on the support of the instrument (the 3x3 experimental design), but one that is within the control of the researcher. Second, it derives identification from assumptions on selection into treatment (with the exception of monotonicity of local average treatment response, which I discuss below), rather than assumptions on outcomes. Third, imposing a strong assumption on the support of the instrument (cross-randomized prices) enables a testable, (often) empirically reasonable, and economically interpretable assumption on selection to suffice for partial identification of LATE-2: no income effects on treatment demand.

In these three regards, the results in this section are most closely related to work that establishes identification of treatment effects in models motivated by sequential, rather than simultaneous, treatment choice (Arteaga, 2023; Humphries et al., 2023; Kamat et al., 2024). Their identification assumptions can be derived under an alternative two-stage experimental design: individuals are first offered a random price for the option to choose between a and b (with outside option 0), and then conditional on that choice, are offered a random price for b (with outside option a). In sharp contrast to the simultaneous design analyzed in this paper, monotonicity (Imbens & Angrist, 1994) is sufficient for identification of a LATE of b relative to a in the sequential design (Heckman & Pinto, 2018; Arteaga, 2023). However, LATE-2 is not necessarily informatively partially identified, as the distribution of outcomes under 0 for 0-to-Always a, 0-to-Complier a-to-b, and 0-to-Always b compliers cannot be separated. Whether this tradeoff suggests the choice of simultaneous or sequential price offers will depend on aspects of the empirical context faced by the researcher.

Section 4.2.3 establishes a formidable barrier to identification of LATE-2: even with full knowledge of treatment demand, discrete variation in prices of multiple treatments is insufficient to guarantee informative bounds on LATE-2. One contribution of this section is to propose the assumption of monotonicity of local average treatment response ("MLATR"), a multidimensional generalization of monotone treatment selection (Manski & Pepper, 2000), and to establish that the bounds on LATE-2 under MLATR asymptotically converge to the MTE with respect to the assigned price increases in the 3x3 experimental design. In light of the nonidentification result in Section 4.2.3, the credibility of MLATR, or an alternative restriction on selection on outcomes, is essential to the empirical applicability of the theoretical results in this paper.

5 Conclusion

Identification of effects of multiple treatments with instrumental variables brings challenges not present in the case of a binary treatment (Imbens & Angrist, 1994): monotonicity and instrument independence and exclusion are no longer sufficient to identify even average flows of individuals between treatments. Recent work has derived identification of treatment effects under parsimonious and testable generalizations of the monotonicity assumption on choice behavior: unordered partial monotonicity and either identical compliers (Mountjoy, 2022) or an additive random utility model (Lee & Salanié, 2018). I extend and bridge identification results across these papers: I show these assumptions are equivalent to no defiers, analogously to Vytlacil (2002), and that treatment effects are identified for individuals indifferent between any set of treatments.

Principally, this paper establishes partial identification of a local average treatment effect for two treatments for a common set of compliers under the 3x3 experimental design cross randomizing zero, one, and two unit increases in the price of two treatments. This experimental design, and the associated estimator, may find applications in contexts in which multidimensional selection into multiple treatments is a first order concern.

This paper does not develop an estimator or a statistical test with continuous instruments. Conditional on estimation of additive utility indices and the distribution of unobserved heterogeneity, results from Kline & Walters (2016) suggest control function approaches can be used to recover the marginal treatment effect. Developing statistical tests of the additive random utility model, and developing estimators of additive utility indices and the distribution of unobserved heterogeneity without parametric restrictions on their functional forms, are important directions for future work.

The conditional equivalence of the assumptions no defiers, identical compliers, and the additive random utility model, and the possibility of uninformative bounds with discrete instruments absent additional assumptions, underscores the difficulty of both relaxing these assumptions and identifying treatment effects. Absent restrictions on heterogeneity of individual treatment responses, unordered partial monotonicity and instrument independence and exclusion are sufficient to identify treatment effects with large instrument support (Heckman et al., 2008), or to bound treatment effects with discrete instruments under additional restrictions on selection on outcomes (Kamat et al., 2023; Lee & Salanié, 2023). This paper proposes monotonicity of local average treatment response as a complementary restriction on outcomes alongside no defiers; however, alternative stronger restrictions on selection on outcomes, coupled with weaker restrictions on selection heterogeneity than no defiers, may enable tighter (and perhaps more credible in a given context) bounds on treatment effects while retaining falsifiability.

References

Abadie, A. (2002). Bootstrap tests for distributional treatment effects in instrumental variable models. *Journal of the American statistical Association*, 97(457), 284–292.

Agness, D. & Getahun, T. (2024). Housing and human capital: Condominiums in ethiopia.

Allcott, H. & Taubinsky, D. (2015). Evaluating behaviorally motivated policy: Experimental evidence from the lightbulb market. *American Economic Review*, 105(8), 2501–38.

Allen, R. & Rehbeck, J. (2019). Identification with additively separable heterogeneity. *Econometrica*, 87(3), 1021–1054.

Arteaga, C. (2023). Parental incarceration and children's educational attainment. *Review of Economics and Statistics*, 105(6), 1394–1410.

Behaghel, L., Crépon, B., & Gurgand, M. (2013). Robustness of the encouragement design in a two-treatment randomized control trial.

Bergman, P., Chetty, R., DeLuca, S., Hendren, N., Katz, L. F., & Palmer, C. (2019). Creating moves to opportunity: Experimental evidence on barriers to neighborhood choice.

Berkouwer, S. B. & Dean, J. T. (2022). Credit, attention, and externalities in the adoption of energy efficient technologies by low-income households. *American Economic Review*, 112(10), 3291–3330.

Bhattacharya, D. (2023). Integrability and identification in multinomial choice models.

Bhuller, M. & Sigstad, H. (2023). 2sls with multiple treatments.

Buchinsky, M., Gertler, P., & Pinto, R. (2024). The economics of monotonicity conditions: Exploring choice incentives in iv models.

Christian, P., Kondylis, F., Loeser, J., Zwager, A., & Glover, S. (2023). Impact evaluation of farmer field schools and evouchers in Mozambique. *AEA RCT Registry*.

Feller, A., Grindal, T., Miratrix, L., & Page, L. C. (2016). Compared to what? variation in the impacts of early childhood education by alternative care type. *The Annals of Applied Statistics*, 10(3).

Heckman, J. J. & Pinto, R. (2018). Unordered monotonicity. *Econometrica*, 86(1), 1–35.

Heckman, J. J., Urzua, S., & Vytlacil, E. (2008). Instrumental variables in models with multiple outcomes: The general unordered case. *Annals of Economics and Statistics*, (pp. 151–174).

Heckman, J. J. & Vytlacil, E. (2005). Structural equations, treatment effects, and econometric policy evaluation. *Econometrica*, 73(3), 669–738.

Hoderlein, S., Holzmann, H., Kasy, M., & Meister, A. (2017). Corrigendum: Instrumental variables with unrestricted heterogeneity and continuous treatment. *The Review of Economic Studies*, 84(2), 964–968.

Hull, P. (2018). Isolateing: Identifying counterfactual-specific treatment effects with cross-stratum comparisons. *Available at SSRN 2705108*.

Humphries, J. E., Ouss, A., Stavreva, K., Stevenson, M. T., & van Dijk, W. (2023). Conviction, incarceration, and recidivism: Understanding the revolving door.

Imbens, G. W. & Angrist, J. D. (1994). Identification and estimation of local average treatment effects. *Econometrica*, 62(2), 467–475.

Kamat, V., Norris, S., & Pecenco, M. (2023). Identification in multiple treatment models under discrete variation.

Kamat, V., Norris, S., & Pecenco, M. (2024). Conviction, incarceration, and policy effects in the criminal justice system.

Kirkeboen, L. J., Leuven, E., & Mogstad, M. (2016). Field of study, earnings, and self-selection. *The Quarterly Journal of Economics*, 131(3), 1057–1111.

Kitagawa, T. (2015). A test for instrument validity. *Econometrica*, 83(5), 2043–2063.

Kline, P. & Tartari, M. (2016). Bounding the labor supply responses to a randomized welfare experiment: A revealed preference approach. *American Economic Review*, 106(4), 972–1014.

Kline, P. & Walters, C. R. (2016). Evaluating public programs with close substitutes: The case of head start. *The Quarterly Journal of Economics*, 131(4), 1795–1848.

Lee, S. & Salanié, B. (2018). Identifying effects of multivalued treatments. *Econometrica*, 86(6), 1939–1963.

Lee, S. & Salanié, B. (2023). Filtered and unfiltered treatment effects with targeting instruments.

Manski, C. F. (1990). Nonparametric bounds on treatment effects. *The American Economic Review*, 80(2), 319–323.

Manski, C. F. & Pepper, J. V. (2000). Monotone instrumental variables: With an application to the returns to schooling. *Econometrica*, 68(4), 997–1010.

McFadden, D. (1981). Econometric models of probabilistic choice. *Structural analysis of discrete data with econometric applications*, 198272.

Mogstad, M., Santos, A., & Torgovitsky, A. (2018). Using instrumental variables for inference about policy relevant treatment parameters. *Econometrica*, 86(5), 1589–1619.

Mogstad, M., Torgovitsky, A., & Walters, C. R. (2021). The causal interpretation of two-stage least squares with multiple instrumental variables. *American Economic Review*, 111(11), 3663–3698.

Mountjoy, J. (2022). Community colleges and upward mobility. *American Economic Review*, 112(8), 2580–2630.

Navjeevan, M. & Pinto, R. (2022). Ordered, unordered and minimal monotonicity criteria.

Pinto, R. (2022). Beyond intention to treat: Using the incentives in moving to opportunity to identify neighborhood effects.

Rose, E. K. & Shem-Tov, Y. (2023). On recoding ordered treatments as binary indicators. *arXiv preprint arXiv:2111.12258*.

Vytlacil, E. (2002). Independence, monotonicity, and latent index models: An equivalence result. *Econometrica*, 70(1), 331–341.

A Appendix

A.1 Assumption UPM ⇒ Assumption TRUM

Proof. Fix $i \in \mathcal{I}$ such that Assumption UPM holds. Suppose the support of $D_i(\cdot)$ is equal to $\mathcal{K}$, that is for each treatment $d \in \mathcal{K}$ there is a value of the instrument that induces i to choose d. Represent the interval support of the instrument $\mathcal{Z} \equiv \bigtimes_{k=1}^{K} \mathcal{Z}_k$, where $\mathcal{Z}_k \equiv [\underline{Z}_k, \overline{Z}_k]$. Assume that $D_i(\underline{Z}) = 1$; this assumption is without loss of generality as, by Assumption UPM, were $D_i(\underline{Z}) = 0$, then $D_i(z) = 0$ for all $z \in \mathcal{Z}$ contradicting the assumption that $D_i(\cdot)$ has full support.

I will show that Assumption TRUM holds. The proof will be constructive. As the resulting utilities $V_{id}(z)$ for $d \in \mathcal{K}$ are ordinal, we can transform them so they take values in $[-1, 1]$. Therefore, if the support of $D_i(\cdot)$ is not equal to $\mathcal{K}$, then for any treatments that i does not select for any value of $z \in \mathcal{Z}$, set their associated utilities equal to -2. Assumption UPM then implies Assumption TRUM.

I proceed in three steps:

- **Willigness-to-pay for** k: I show there exists z^*_{ik} for each $k \in \mathcal{K} \setminus \{0\}$, such that $D_i(z) = 0$ if and only if $z_k > z^*_{ik}$ for all $k \in \mathcal{K} \setminus \{0\}$, and $D_i(z) = k$ only if $z_k < z^*_{ik}$.
- **Equivalent variation of option to buy** k **relative to choosing** 1: For all $k \in \mathcal{K} \setminus \{0, 1\}$, I show there exists $\overline{z}_{i1k}(z_k)$ such that $\overline{z}_{i1k}(z^*_{ik}) = z^*_{i1}$, $\overline{z}_{i1k}(z_k)$ is increasing in z_k, $D_i(z) = 1$ only if $z_1 < \overline{z}_{i1k}(z_k)$, and $D_i(z) = k$ only if $z_1 > \overline{z}_{i1k}(z_k)$.
- **From equivalent variations to utility indices**: I then construct $\mu_{ik}(z_k)$ for each $k \in \mathcal{K} \setminus \{0\}$ satisfying Assumption TRUM.

Interpreting instruments as prices, the first step corresponds to showing that each individual has a maximum willingness-to-pay z^*_{ik} for treatment k, and the individual doesn't buy anything if and only if all prices are above their maximum willingness-to-pays. The second step corresponds to constructing the equivalent variation for the opportunity to buy k at price z_k instead of good 1 at price z_1: this equivalent variation is equal to $z_1 - \overline{z}_{i1k}(z_k)$, that is the amount that leaves individual i indifferent between choosing 1 and receiving that amount (paying $\overline{z}_{i1k}(z_k)$), and choosing k (paying z_k). The third step uses equivalent variations to construct utility indices: individual i

chooses good k over good j only if the equivalent variation of the option to choose k, relative to choosing good 1, is larger than that of j.

Construction of z^*_{ik} For all $k \in \mathcal{K} \setminus \{0\}$, let

$$z^*_{ik} \equiv \max_{z \in \mathcal{Z}: D_i(z)=k} z_k$$

By construction, $D_i(z) = k$ only if $z_k < z^*_{ik}$.

It therefore holds that $D_i(z) = 0$ if $z_k > z^*_{ik}$ for all $k \in \mathcal{K} \setminus \{0\}$.

Suppose, for contradiction, there exists $k \in \mathcal{K} \setminus \{0\}$ and $z \equiv (z_k, z_{-k})$ such that $z_k < z^*_{ik}$, but $D_i(z) = 0$. By definition of z^*_{ik}, there exists $z' \equiv (z'_k, z'_{-k})$ such that $z'_k > z_k$ and $D_i(z') = k$. Define z''_{-k} to be the element-wise maximum of z_{-k} and z'_{-k}. As $\mathcal{Z}$ is an interval of $\mathbb{R}^K$, $(z_k, z''_{-k}), (z'_k, z''_{-k}) \in \mathcal{Z}$. By Assumption UPM, $D_i(z_k, z''_{-k}) = 0$ and $D_i(z'_k, z''_{-k}) = k$, as increasing z_{-k} and z'_{-k}, respectively, cannot cause any individuals to shift out of 0 nor out of k. However, we then have $z'_k > z_k$, $D_i(z'_k, z''_{-k}) = k$, but $D_i(z_k, z''_{-k}) = 0$, contradicting Assumption UPM. Therefore, $D_i(z) = 0$ implies that $z_k \geq z^*_{ik}$ for all $k \in \mathcal{K} \setminus \{0\}$.

Construction of $\overline{z}_{i1k}(z_k)$ Let

$$\overline{z}_{i1k}(z_k) \equiv \max_{z' \in \mathcal{Z}: D_i(z')=1, z'_k \leq z_k} z'_1$$

$\overline{z}_{i1k}(z_k)$ exists, as $D_i(z_k, \underline{Z}_{-k}) = 1$ by Assumption UPM, as $D_i(\underline{Z}) = 1$ and $z_k \geq \underline{Z}_k$.

By construction, $\overline{z}_{i1k}(z_k)$ is increasing in z_k, as it takes the maximum of a function over an expanding set. Similarly, $\overline{z}_{i1k}(z_k) \leq z^*_{i1}$.

Fix $k \in \mathcal{K} \setminus \{0, 1\}$, $z_k > z^*_{ik}$. Choose any $z' \in \mathcal{Z}$ such that $D_i(z') = 0$; therefore, $z'_\ell > z^*_{i\ell}$ for all $\ell \in \mathcal{K} \setminus \{0\}$. Note that $D_i(z'_1, z_k, z'_{-\{1,k\}}) = 0$, as it remains that $z_k > z^*_{ik}$ and we have not changed any other elements of z'. In turn, note that $D_i(z^*_{i1}, z_k, z'_{-\{1,k\}}) = 1$, as we have decreased z'_1 to z^*_{i1} (so $D_i(z^*_{i1}, z_k, z'_{-\{1,k\}}) \neq 0$) without changing any other elements of z'. Therefore, $\overline{z}_{i1k}(z_k) = z^*_{i1}$ for all $z_k > z^*_{ik}$.

By construction, $D_i(z) = 1$ only if $z_1 \leq \overline{z}_{i1k}(z_k)$.

Suppose, for contradiction, there exists z such that $D_i(z) = k$ and $z_1 < \overline{z}_{i1k}(z_k)$. Find z' such that $z'_1 = \overline{z}_{i1k}(z_k)$, $z'_k \leq z_k$, $D_i(z') = 1$; such z' exists by definition of $\overline{z}_{i1k}(z_k)$. Note that $D_i(z_k, z'_{-k}) = 1$ by Assumption UPM, as increasing z'_k to z_k cannot cause i to shift out of 1. Define z'' by $z''_1 = z_1$, $z''_k = z_k$, and $z''_\ell = \max\{z_\ell, z'_\ell\}$ for all $\ell \in \mathcal{K} \setminus \{0, 1, k\}$; $D_i(z'') = k$ by Assumption UPM as $D_i(z) = k$. Define z''' by $z'''_1 = z'_1$, $z'''_k = z_k$, $z'''_\ell = \max\{z_\ell, z'_\ell\}$ for all $\ell \in \mathcal{K} \setminus \{0, 1, k\}$; $D_i(z''') = 1$ by Assumption UPM as $z_k \geq z'_k$ and $D_i(z') = 1$. z'' and z''' are identical, except $z'''_1 = \overline{z}_{i1k}(z_k) > z_1 = z''_1$, contradicting Assumption UPM. Therefore, $D_i(z) = k$ only if $z_1 \geq \overline{z}_{i1k}(z_k)$.

Construction of $\mu_{ik}(z_k)$ Let

$$\mu_{i1}(z_1) \equiv z_1 - z^*_{i1}$$

and, for all $k \in \mathcal{K} \setminus \{0, 1\}$,

$$\mu_{ik}(z_k) \equiv \mathbf{1}\{z_k \leq z^*_{ik}\} \left(\overline{z}_{i1k}(z_k) - z^*_{i1}\right) + \mathbf{1}\{z_k > z^*_{ik}\} \left(z_k - z^*_{ik}\right)$$

I set $U_{ik} = 0$ for all $k \in \mathcal{K} \setminus \{0\}$.

Note that $\mu_{ik}(z_k)$ is an increasing function of z_k, as z_k and $\overline{z}_{i1k}(z_k)$ are both increasing functions of z_k, $z_k - z^*_{ik} > 0$ whenever $z_k > z^*_{ik}$, and $\overline{z}_{i1k}(z_k) - z^*_{i1} \leq 0$.

For Assumption TRUM, it therefore suffices to show that, for any $k \in \mathcal{K}$, $D_i(z) = k$ if and only if $\mu_{ik}(z_k) < \mu_{i\ell}(z_\ell)$ for all $\ell \in \mathcal{K} \setminus \{k\}$, with the abuse of notation that $\mu_{i0}(z_0) = 0$.

First, note that $D_i(z) = 0$ if and only if $z_k \geq z^*_{ik}$ for all $k \in \mathcal{K} \setminus \{0\}$, which holds if and only if $\mu_{ik}(z_k) \leq 0$ for all $k \in \mathcal{K} \setminus \{0\}$.

Second, note that $D_i(z) = 1$ if and only if $z_1 < z^*_{i1}$ and $z_1 \leq \overline{z}_{i1k}(z_k)$ for all $k \in \mathcal{K} \setminus \{0\}$. These hold if and only if $\mu_{i1}(z_1) < 0$ and $\mu_{i1}(z_1) \leq \mu_{ik}(z_k)$ for all $k \in \mathcal{K} \setminus \{0\}$, respectively.

It remains to show that $D_i(z) = k$ if and only if $\mu_{ik}(z_k) < \mu_{i\ell}(z_\ell)$ for all $\ell \in \mathcal{K} \setminus \{k\}$, for all $k \in \mathcal{K} \setminus \{0, 1\}$.

Proof of $D_i(z) = k \Leftarrow \mu_{ik}(z_k) < \mu_{i\ell}(z_\ell)$ I proceed in three steps: I show $D_i(z) \neq 0$, $D_i(z) \neq 1$, and $D_i(z) \neq \ell$ for all $\ell \in \mathcal{K} \setminus \{0, 1, k\}$. Therefore, $D_i(z) = k$.

First, by assumption, $\mu_{ik}(z_k) < 0$. $\mu_{ik}(z_k) > 0$ whenever $z_k > z^*_{ik}$, and therefore $z_k < z^*_{ik}$. Therefore, $D_i(z) \neq 0$ by definition of z^*_{ik}.

Second, as $z_k < z^*_{ik}$, $\mu_{ik}(z_k) = \overline{z}_{i1k}(z_k) - z^*_{i1}$. By assumption, $\mu_{ik}(z_k) < \mu_{i1}(z_1)$;

substituting definitions into the inequality yields $\overline{z}_{i1k}(z_k) - z^*_{i1} < z_1 - z^*_{i1}$, or $\overline{z}_{i1k}(z_k) < z_1$. By definition of $\overline{z}_{i1k}(z_k)$, then $D_i(z) \neq 1$.

Third, suppose for contradiction that $D_i(z) = \ell$ for $\ell \in \mathcal{K} \setminus \{0, 1, k\}$. Therefore, $z_\ell < z^*_{i\ell}$, and $\mu_{i\ell}(z_\ell) = \overline{z}_{i1\ell}(z_\ell) - z^*_{i1}$. By assumption, $\mu_{ik}(z_k) < \mu_{i\ell}(z_\ell)$, and therefore $\overline{z}_{i1k}(z_k) < \overline{z}_{i1\ell}(z_\ell)$. By definition, this implies that there exists z' such that $z'_1 \leq \overline{z}_{i1\ell}(z_\ell)$, $z'_1 > \overline{z}_{i1k}(z_k)$, $z'_\ell = z_\ell$, and $D_i(z') = 1$. Set z'_k equal to z_k; by Assumption UPM, either $D_i(z') = 1$ or $D_i(z') = k$. As $z'_1 > \overline{z}_{i1k}(z_k)$, $D_i(z') \neq 1$, and therefore $D_i(z') = k$. Set z_j and z'_j equal to $\max\{z_j, z'_j\}$ for all $j \in \mathcal{K} \setminus \{0, 1, k, \ell\}$. By Assumption UPM, $D_i(z) = \ell$ and $D_i(z') = k$. Finally, set $z'_1 = z_1$; either $D_i(z') = k$ or $D_i(z') = 1$, but this is a contradiction as $z' = z$. Therefore, $D_i(z) \neq \ell$.

Proof of $D_i(z) = k \Rightarrow \mu_{ik}(z_k) < \mu_{i\ell}(z_\ell)$ I proceed in three steps: I show $D_i(z) = k$ implies $\mu_{ik}(z_k) \leq 0$, $\mu_{ik}(z_k) \leq \mu_{i1}(z_1)$, and $\mu_{ik}(z_k) \leq \mu_{i\ell}(z_\ell)$.

First, $D_i(z) = k$ implies that $z_k < z^*_{ik}$. Therefore, $\mu_{ik}(z_k) = \overline{z}_{i1k}(z_k) - z^*_{i1}$, and therefore $\mu_{ik}(z_k) \leq 0$.

Second, $D_i(z) = k$ implies that $z_1 > \overline{z}_{i1k}(z_k)$. Therefore $z_1 - z^*_{i1} > \overline{z}_{i1k}(z_k) - z^*_{i1}$, and substituting yields $\mu_{ik}(z_k) < \mu_{i1}(z_1)$.

Third, fix $\ell \in \mathcal{K} \setminus \{0, 1, k\}$. Suppose for contradiction that $\overline{z}_{i1k}(z_k) > \overline{z}_{i1\ell}(z_\ell)$. Then there exists z' such that $z'_1 \leq \overline{z}_{i1k}(z_k)$, $z'_1 > \overline{z}_{i1\ell}(z_\ell)$, $z'_k = z_k$, and $D_i(z') = 1$. Set $z'_\ell = z_\ell$; by Assumption UPM either $D_i(z') = 1$ or $D_i(z') = \ell$. As $z'_1 > \overline{z}_{i1\ell}(z_\ell)$, $D_i(z') \neq 1$, and therefore $D_i(z') = \ell$. Set z_j and z'_j equal to $\max\{z_j, z'_j\}$ for all $j \in \mathcal{K} \setminus \{0, 1, k, \ell\}$. By Assumption UPM, $D_i(z) = k$, and $D_i(z') = \ell$. Finally, set $z'_1 = z_1$; either $D_i(z') = \ell$ or $D_i(z') = 1$, but this is a contradiction as $z' = z$. Therefore, $\overline{z}_{i1k}(z_k) \leq \overline{z}_{i1\ell}(z_\ell)$, and therefore $\mu_{ik}(z_k) \leq \mu_{i\ell}(z_\ell)$. □

A.2 Assumption IC ⇒ Homogeneous instrument sensitivity

Proof. First, I characterize the set of values of U_i conditional on μ_i that correspond to $D_i(z'_k, z_{-k}) = k, D_i(z_k, z_{-k}) = \ell$. Note $U_{i\ell} - \mu_{i\ell}(z_\ell) \geq \max_{\ell' \in \mathcal{K} \setminus \{k, \ell\}} U_{i\ell'} - \mu_{i\ell'}(z_{\ell'})$; the values of U_i satisfying this inequality are fixed with respect to z'_k, and include all possible values of U_{ik}. Therefore represent the values of the latent indices $U_{i,-k}$ satisfying this inequality as $\mathcal{U}_{\ell,-k}(z_{-k}, \mu_i) \equiv \{u_{-k} : u_\ell - \mu_{i\ell}(z_\ell) \geq \max_{\ell' \in \mathcal{K} \setminus \{k, \ell\}} u_{\ell'} - \mu_{i\ell'}(z_{\ell'})\}$. Note, by construction, $D_i(z_k, z_{-k}) = \ell$ implies that $U_{i,-k} \in \mathcal{U}_{\ell,-k}(z_{-k}, \mu_i)$. Further, as $D_i(z'_k, z_{-k}) = k$, $U_{ik} - \mu_{ik}(z'_k) \geq U_{i\ell} - \mu_{i\ell}(z_\ell) \geq U_{ik} - \mu_{ik}(z_k)$, and therefore $U_{ik} \in [U_{i\ell} - \mu_{i\ell}(z_\ell) + \mu_{ik}(z'_k), U_{i\ell} - \mu_{i\ell}(z_\ell) + \mu_{ik}(z_k)]$.

I let $f_{k|-k}(\cdot|U_{i,-k}, \mu_i)$ denote the conditional density of U_{ik} given $U_{i,-k}$. I let $f_{-k}(\cdot|\mu_i)$ denote the density of $U_{i,-k}$. I proceed in 4 steps below. I begin with the left side of the equality in Assumption IC, the limit as z'_k increases to z_k of the expectation of $y(V_i)$ given i chooses k at (z'_k, z_{-k}) but chooses ℓ at (z_k, z_{-k}).

First, I express the conditional expectation as the ratio of integrals, of $y(u - m(\cdot))f(u|m)$ and $f(u|m)$, respectively, over the set of values u satisfying the above treatment choices, and with respect to the probability measure F over μ_i. I factor the integration, and the density f, into integration over u_k conditional on u_{-k}, and over u_{-k}. The denominator is non-zero, as it is strictly positive for each $\mu_i = m$, as the density f is strictly positive (Assumption TRUM.2) and $\mu'_{ik}(z_k)$ is strictly positive (Assumption TRUM.3).

Second, I note the numerator and denominator of the ratio both converge to zero, as y (Assumption IC) and f (Assumption TRUM.2) are both bounded and the limits of integration for u_k conditional on u_{-k} converge. I therefore apply L'Hopital's rule to evaluate the limit, replacing the numerator and the denominator with their derivatives with respect to z'_2 evaluated at $z'_2 = z_2$.

Third, I exchange the order of differentiation and integration, and fourth, I evaluate the derivative. The derivative involves changing the bounds of an integral in the limit as the bounds converge, and is well defined as the conditional density, or its product with y, being integrated is continuous. The resulting term after taking derivatives is bounded, as y, f, and μ_{ik} (by Assumption TRUM.1) are bounded, allowing exchanging differentiation and integration in the third step.

$$
\begin{aligned}
&\lim_{z_k'\uparrow z_k} \mathbf{E}[y(V_i)|D_i(z_k', z_{-k}) = k, D_i(z_k, z_{-k}) = \ell] \\
&= \lim_{z_k'\uparrow z_k} \frac{\int\int_{\mathcal{U}_{\ell,-k}(z_{-k},\mu_i)} \int_{u_\ell-\mu_{i\ell}(z_\ell)+\mu_{ik}(z_k')}^{u_\ell-\mu_{i\ell}(z_\ell)+\mu_{ik}(z_k)} y(u-m(\cdot)) f_{k|-k}(u_k|u_{-k},m) du_k f_{-k}(u_{-k}|m) du_{-k} dF(m)}{\int\int_{\mathcal{U}_{\ell,-k}(z_{-k},\mu_i)} \int_{u_\ell-\mu_{i\ell}(z_\ell)+\mu_{ik}(z_k')}^{u_\ell-\mu_{i\ell}(z_\ell)+\mu_{ik}(z_k)} f_{k|-k}(u_k|u_{-k},m) du_k f_{-k}(u_{-k}|m) du_{-k} dF(m)} \\
&= \frac{\frac{d}{dz_k'}\int\int_{\mathcal{U}_{\ell,-k}(z_{-k},\mu_i)} \int_{u_\ell-\mu_{i\ell}(z_\ell)+\mu_{ik}(z_k')}^{u_\ell-\mu_{i\ell}(z_\ell)+\mu_{ik}(z_k)} y(u-m(\cdot)) f_{k|-k}(u_k|u_{-k},m) du_k f_{-k}(u_{-k}|m) du_{-k} dF(m)\Big|_{z_k'=z_k}}{\frac{d}{dz_k'}\int\int_{\mathcal{U}_{\ell,-k}(z_{-k},\mu_i)} \int_{u_\ell-\mu_{i\ell}(z_\ell)+\mu_{ik}(z_k')}^{u_\ell-\mu_{i\ell}(z_\ell)+\mu_{ik}(z_k)} f_{k|-k}(u_k|u_{-k},m) du_k f_{-k}(u_{-k}|m) du_{-k} dF(m)\Big|_{z_k'=z_k}} \\
&= \frac{\int\int_{\mathcal{U}_{\ell,-k}(z_{-k},\mu_i)} \frac{d}{dz_k'} \int_{u_\ell-\mu_{i\ell}(z_\ell)+\mu_{ik}(z_k')}^{u_\ell-\mu_{i\ell}(z_\ell)+\mu_{ik}(z_k)} y(u-m(\cdot)) f_{k|-k}(u_k|u_{-k},m) du_k\Big|_{z_k'=z_k} f_{-k}(u_{-k}|m) du_{-k} dF(m)}{\int\int_{\mathcal{U}_{\ell,-k}(z_{-k},\mu_i)} \frac{d}{dz_k'} \int_{u_\ell-\mu_{i\ell}(z_\ell)+\mu_{ik}(z_k')}^{u_\ell-\mu_{i\ell}(z_\ell)+\mu_{ik}(z_k)} f_{k|-k}(u_k|u_{-k},m) du_k\Big|_{z_k'=z_k} f_{-k}(u_{-k}|m) du_{-k} dF(m)} \\
&= \frac{\int\int_{\mathcal{U}_{\ell,-k}(z_{-k},\mu_i)} y((u_\ell-\mu_{i\ell}(z_\ell)+\mu_{ik}(z_k), u_{-k}) - m(\cdot)) \mu_{ik}'(z_k) f((u_\ell-\mu_{i\ell}(z_\ell)+\mu_{ik}(z_k), u_{-k})|m) du_{-k} dF(m)}{\int\int_{\mathcal{U}_{\ell,-k}(z_{-k},\mu_i)} \mu_{ik}'(z_k) f((u_\ell-\mu_{i\ell}(z_\ell)+\mu_{ik}(z_k), u_{-k})|m) du_{-k} dF(m)}
\end{aligned} \tag{A1}
$$

I apply the same sequence of steps for the right hand side of the equality of Assumption IC. I then apply one additional step: I substitute $u_k = u_\ell - \mu_{i\ell}(z_\ell) + \mu_{ik}(z_k)$ inside the integrals. This sequence of steps, and this change of variable, yields

$$
\begin{aligned}
&\lim_{z_\ell'\downarrow z_\ell} \mathbf{E}[y(V_i)|D_i(z_\ell', z_{-\ell}) = k, D_i(z_\ell, z_{-\ell}) = \ell] \\
&= \frac{\int\int_{\mathcal{U}_{k,-\ell}(z_{-\ell},\mu_i)} y((u_k-\mu_{ik}(z_k)+\mu_{i\ell}(z_\ell), u_{-\ell}) - m(\cdot)) \mu_{i\ell}'(z_\ell) f((u_k-\mu_{ik}(z_k)+\mu_{i\ell}(z_\ell), u_{-\ell})|m) du_{-\ell} dF(m)}{\int\int_{\mathcal{U}_{k,-\ell}(z_{-\ell},\mu_i)} \mu_{i\ell}'(z_\ell) f((u_k-\mu_{ik}(z_k)+\mu_{i\ell}(z_\ell), u_{-\ell}) du_{-\ell} dF(m)} \\
&= \frac{\int\int_{\mathcal{U}_{\ell,-k}(z_{-k},\mu_i)} y((u_\ell-\mu_{i\ell}(z_\ell)+\mu_{ik}(z_k), u_{-k}) - m(\cdot)) \mu_{i\ell}'(z_\ell) f((u_\ell-\mu_{i\ell}(z_\ell)+\mu_{ik}(z_k), u_{-k})|m) du_{-k} dF(m)}{\int\int_{\mathcal{U}_{\ell,-k}(z_{-k},\mu_i)} \mu_{i\ell}'(z_\ell) f((u_\ell-\mu_{i\ell}(z_\ell)+\mu_{ik}(z_k), u_{-k})|m) du_{-k} dF(m)}
\end{aligned} \tag{A2}
$$

Assumption IC implies Equations A1 and A2 are equal; these equations are integrals of the density of "marginal" individuals (those for whom $U_{i\ell} - \mu_{i\ell}(z_\ell) = U_{ik} - \mu_{ik}(z_k)$), with "marginal" individual i weighted by $\mu_{ik}'(z_k)$ and $\mu_{i\ell}'(z_\ell)$. While the equality characterizing marginal individuals, $U_{i\ell} - \mu_{i\ell}(z_\ell) = U_{ik} - \mu_{ik}(z_k)$, is well defined, the expectation $\mathbf{E}[y(V_i)|U_{i\ell} - \mu_{i\ell}(z_\ell) = U_{ik} - \mu_{ik}(z_k)]$ is not necessarily well defined, as the conditioning set has zero measure (Hoderlein et al., 2017). Consequentially, in Equations A1 and A2, the weights on individuals are proportional to the rate at which the k-to-ℓ complier fraction approaches zero in the limit, which is proportional to responsiveness $\mu'_{ik}(z_k)$ and $\mu'_{i\ell}(z_\ell)$, respectively. Equality for all possible $y(\cdot)$ and z requires these weighting schemes are identical, and therefore $\frac{\mu'_{ik}(z_k)}{\mu'_{i\ell}(z_\ell)} = \frac{\mu'_{jk}(z_k)}{\mu'_{j\ell}(z_\ell)}$ for all $i, j \in \mathcal{I}$, $z \in \mathcal{Z}$. This is equivalent to homogeneous instrument sensitivity. $\square$

A.3 Proposition MTR

Proof. I proceed in four steps. First, I briefly prove Equation 3; while this replicates the result from Allen & Rehbeck (2019) and Bhattacharya (2023), I present it here as I apply a similar approach to the proof of generalizations of Equations 4 and 5. Second, third, and fourth, I derive generalizations of Equations 4 and 5 in three cases: when $k = 0$, when $k \neq 0$ and $0 \in \mathcal{K}_0$, and when $k \neq 0$ and $0 \notin \mathcal{K}_0$.

Equation 3 By Assumption TRUM and ARUM,

$$P_k(z) = \int_{\mu_k(z_k)}^{\infty} \int_{-\infty}^{u_k - \mu_k(z_k) + \mu_\ell(z_\ell)} \int_{-\infty}^{u_k - \mu_k(z_k) + \mu_{-\{k,\ell\}}(z_{-\{k,\ell\}})} f(u) du_{-\{k,\ell\}} du_\ell du_k \quad \text{(A3)}$$

Equation A3 integrates the density of U_i over the values of u that solve the inequalities $u_k - \mu_k(z_k) \geq 0$ and $u_k - \mu_k(z_k) \geq u_\ell - \mu_{\ell'}(z_{\ell'})$ for all $\ell' \in \mathcal{K} \setminus \{0, k\}$. I divide these latter inequalities into one for ℓ, and the remainder for $\ell' \in \mathcal{K} \setminus \{0, k, \ell\}$.

To differentiate with respect to z_ℓ, I first factor the density

$$f(u) = f_k(u_k) f_{\ell|k}(u_\ell | u_k) f_{-\{k,\ell\}|k,\ell}(u_{-\{k,\ell\}} | u_k, u_\ell)$$

I then exchange the order of differentiation and the first integral, and apply differentiation. As a final step, I apply the change of variable $u_k = u_\ell + \mu_k(z_k) - \mu_\ell(z_\ell)$ to the integral with respect to u_k.

$$
\begin{aligned}
&\frac{\partial}{\partial z_\ell}P_k(z)\\
&=\frac{\partial}{\partial z_\ell}\int_{\mu_k(z_k)}^{\infty} f_k(u_k)\int_{-\infty}^{u_k-\mu_k(z_k)+\mu_\ell(z_\ell)} f_{\ell|k}(u_\ell|u_k)\int_{-\infty}^{u_k-\mu_k(z_k)+\mu_{-\{k,\ell\}}(z_{-\{k,\ell\}})} f_{-\{k,\ell\}|k,\ell}(u_{-\{k,\ell\}}|u_k,u_\ell)du_{-\{k,\ell\}}du_\ell du_k\\
&=\int_{\mu_k(z_k)}^{\infty} f_k(u_k)\frac{\partial}{\partial z_\ell}\int_{-\infty}^{u_k-\mu_k(z_k)+\mu_\ell(z_\ell)} f_{\ell|k}(u_\ell|u_k)\int_{-\infty}^{u_k-\mu_k(z_k)+\mu_{-\{k,\ell\}}(z_{-\{k,\ell\}})} f_{-\{k,\ell\}|k,\ell}(u_{-\{k,\ell\}}|u_k,u_\ell)du_{-\{k,\ell\}}du_\ell du_k\\
&=\mu_\ell'(z_\ell)\int_{\mu_k(z_k)}^{\infty} f_{k,\ell}(u_k,u_k-\mu_k(z_k)+\mu_\ell(z_\ell))\int_{-\infty}^{u_k-\mu_k(z_k)+\mu_{-\{k,\ell\}}(z_{-\{k,\ell\}})} f_{-\{k,\ell\}|k,\ell}(u_{-\{k,\ell\}}|u_k,u_k-\mu_k(z_k)+\mu_\ell(z_\ell))du_{-\{k,\ell\}}du_k\\
&=\mu_\ell'(z_\ell)\int_{\mu_\ell(z_\ell)}^{\infty} f_{k,\ell}(u_\ell-\mu_\ell(z_\ell)+\mu_k(z_k),u_\ell)\int_{-\infty}^{u_\ell-\mu_\ell(z_\ell)+\mu_{-\{k,\ell\}}(z_{-\{k,\ell\}})} f_{-\{k,\ell\}|k,\ell}(u_{-\{k,\ell\}}|u_\ell-\mu_\ell(z_\ell)+\mu_k(z_k),u_\ell)du_{-\{k,\ell\}}du_\ell
\end{aligned}
\tag{A4}
$$

Excluding the last step in Equation A4, applying the same steps to $\frac{\partial}{\partial z_k}\mathbf{E}[D_{i\ell}(z)]$ yields the same expression as the last line of Equation A4, with the exception that $\mu_\ell'(z_\ell)$ is replaced with $\mu_k'(z_k)$; taking the ratio of these two expressions immediately yields Equation 3.

Generalizations of Equations 4 and 5 when $k = 0$ For $k = 0$, note that

$$P_0(z) = \int_{-\infty}^{\mu(z)} f(u)du \tag{A5}$$

Further, substituting $\mathbf{E}[Y_{i0}|U_i = u] \equiv \mathrm{MTR}_0(u)$,

$$PY_0(z) = \int_{-\infty}^{\mu(z)} \mathrm{MTR}_0(u)f(u)du \tag{A6}$$

Next, choose $\mathcal{K}_0 \subseteq \mathcal{K}$ such that $0 \in \mathcal{K}_0$, and denote $\mathcal{K}_0^* \equiv \mathcal{K}_0 \setminus \{0\}$. Differentiating both sides of Equation A5 with respect to z_k for all $k \in \mathcal{K}_0^* \setminus k$ yields

$$\frac{\partial^{|\mathcal{K}_0|-1}}{\prod_{k\in\mathcal{K}_0^*}\partial z_k}P_0(z) = \left(\prod_{k\in\mathcal{K}_0^*}\mu_k'(z_k)\right)\int_{-\infty}^{\mu_{-\mathcal{K}_0^*}(z_{-\mathcal{K}_0^*})} f(\mu_{\mathcal{K}_0^*}(z_{\mathcal{K}_0^*}),u_{-\mathcal{K}_0^*})du_{-\mathcal{K}_0^*} \tag{A7}$$

Applying the same approach to Equation A6 yields

$$\frac{\partial^{|\mathcal{K}_0|-1}}{\prod_{k\in\mathcal{K}_0^*}\partial z_k}PY_0(z) = \left(\prod_{k\in\mathcal{K}_0^*}\mu_k'(z_k)\right)\int_{-\infty}^{\mu_{-\mathcal{K}_0^*}(z_{-\mathcal{K}_0^*})} \mathrm{MTR}_0(\mu_{\mathcal{K}_0^*}(z_{\mathcal{K}_0^*}),u_{-\mathcal{K}_0^*})f(\mu_{\mathcal{K}_0^*}(z_{\mathcal{K}_0^*}),u_{-\mathcal{K}_0^*})du_{-\mathcal{K}_0^*} \tag{A8}$$

Taking ratios, yields

$$\begin{aligned}\frac{\frac{\partial^{|\mathcal{K}_0|-1}}{\prod_{k\in\mathcal{K}_0^*}\partial z_k}PY_0(z)}{\frac{\partial^{|\mathcal{K}_0|-1}}{\prod_{k\in\mathcal{K}_0^*}\partial z_k}P_0(z)} &= \frac{\int_{-\infty}^{\mu_{-\mathcal{K}_0^*}(z_{-\mathcal{K}_0^*})}\mathrm{MTR}_0(\mu_{\mathcal{K}_0^*}(z_{\mathcal{K}_0^*}),u_{-\mathcal{K}_0^*})f(\mu_{\mathcal{K}_0^*}(z_{\mathcal{K}_0^*}),u_{-\mathcal{K}_0^*})du_{-\mathcal{K}_0^*}}{\int_{-\infty}^{\mu_{-\mathcal{K}_0^*}(z_{-\mathcal{K}_0^*})}f(\mu_{\mathcal{K}_0^*}(z_{\mathcal{K}_0^*}),u_{-\mathcal{K}_0^*})du_{-\mathcal{K}_0^*}} \\ &= \mathbf{E}[\mathrm{MTR}_0(U_i)|U_{i\mathcal{K}_0^*}=\mu_{\mathcal{K}_0^*}(z_{\mathcal{K}_0^*}),U_{i,-\mathcal{K}_0^*}\le\mu_{-\mathcal{K}_0^*}(z_{-\mathcal{K}_0^*})] \\ &= \mathbf{E}[\mathrm{MTR}_0(U_i)|U_i\in\mathcal{U}(\mathcal{K}_0)]\end{aligned} \tag{A9}$$

which establishes identification of the average marginal treatment response when $k=0$ for Proposition MTR. Intuitively, we can identify the mean potential outcome under 0 for individuals indifferent between 0 and any set of treatments $\mathcal{K}_0^*$ as these are exactly the individuals who are pushed into 0 by a small increase in the instrument associated with each treatment in $\mathcal{K}_0^*$.

Lastly, note that when $\mathcal{K}_0=\mathcal{K}$, the right hand side of Equation A9 simplifies to $\mathrm{MTR}_0(\mu(z))$.

Generalizations of Equations 4 and 5 when $k\neq 0$, $0\notin\mathcal{K}_0$ Let $\mathcal{K}_0^*\equiv\mathcal{K}_0\setminus\{k\}$. I apply the steps in Equation A4 to derive

$$\frac{\partial^{|\mathcal{K}_0|-1}}{\prod_{\ell\in\mathcal{K}_0^*}\partial z_\ell}P_k(z)=\left(\prod_{\ell\in\mathcal{K}_0^*}\mu_\ell'(z_\ell)\right)\int_{\mu_k(z_k)}^{\infty}\int_{-\infty}^{u_k-\mu_k(z_k)+\mu_{-\mathcal{K}_0}(z_{-\mathcal{K}_0})}f(u_k-\mu_k(z_k)+\mu_{\mathcal{K}_0^*}(z_{\mathcal{K}_0^*}),u_{-\mathcal{K}_0^*})du_{-\mathcal{K}_0}du_k$$

I take the same $(|\mathcal{K}_0|-1)$th partial derivative of $PY_k(z)$, and take ratios. Noting that

$$\{u:u_k\ge\mu_k(z_k),u_{-\mathcal{K}_0}\le u_k-\mu_k(z_k)+\mu_{-\mathcal{K}_0}(z_{-\mathcal{K}_0}),u_{\mathcal{K}_0^*}=u_k-\mu_k(z_k)+\mu_{\mathcal{K}_0^*}(z_{\mathcal{K}_0^*})\}=\mathcal{U}(\mathcal{K}_0)$$

I derive

$$\frac{\frac{\partial^{|\mathcal{K}_0|-1}}{\prod_{\ell\in\mathcal{K}_0^*}\partial z_\ell}PY_k(z)}{\frac{\partial^{|\mathcal{K}_0|-1}}{\prod_{\ell\in\mathcal{K}_0^*}\partial z_\ell}P_k(z)}=\mathbf{E}[\mathrm{MTR}_k(U_i)|U_i\in\mathcal{U}(\mathcal{K}_0)] \tag{A10}$$

When $|\mathcal{K}_0|=2$, Equation A10 is equivalent to Equation 14 in Mountjoy (2022).

Generalizations of Equations 4 and 5 when $k\neq 0$, $0\in\mathcal{K}_0$ I begin by taking the derivative of both sides of Equation A3 with respect to increasing $\mu_\ell(z_\ell)$ by an identical amount for all $\ell \in \mathcal{K} \setminus \{0\}$.

$$\frac{\partial}{\sum_{\ell \in \mathcal{K}\setminus\{0\}} \mu'_\ell(z_\ell)\partial z_\ell} P_k(z) = \int_{-\infty}^{\mu_{-k}(z_k)} f(\mu_k(z_k), u_{-k}) du_{-k} \tag{A11}$$

Note that in taking this derivative, we implicitly assumed that $\mu'_\ell(z_\ell)/\mu'_k(z_k)$ is identified for all $\ell \in \mathcal{K}$, as established by Equation 3.

Let $\mathcal{K}_0^* \equiv \mathcal{K}_0 \setminus \{0, k\}$, and $\tilde{\mathcal{K}}_0 \equiv \mathcal{K}_0 \setminus \{0\}$. Differentiating with respect to z_ℓ for each $\ell \in \mathcal{K}_0^*$ yields

$$\frac{\partial}{\sum_{\ell \in \mathcal{K}\setminus\{0\}} \mu'_\ell(z_\ell)\partial z_\ell} \frac{\partial^{|\mathcal{K}_0|-2}}{\prod_{\ell \in \mathcal{K}_0^*} \partial z_\ell} P_k(z) = \left(\prod_{\ell \in \mathcal{K}_0^*} \mu'_\ell(z_\ell) \right) \int_{-\infty}^{\mu_{-\tilde{\mathcal{K}}_0}(z_{-\tilde{\mathcal{K}}_0})} f(\mu_{\tilde{\mathcal{K}}_0}(z_{\tilde{\mathcal{K}}_0}), u_{-\tilde{\mathcal{K}}_0}) du_{-\tilde{\mathcal{K}}_0}$$

I take the same $(|\mathcal{K}_0| - 1)$th partial derivative of $PY_k(z)$, and take ratios. Noting that

$$\{u : u_k = \mu_k(z_k), u_{\mathcal{K}_0^*} = \mu_{\mathcal{K}_0^*}(z_{\mathcal{K}_0^*}), u_{-\mathcal{K}_0} \leq \mu_{-\mathcal{K}_0}\} = \mathcal{U}(\mathcal{K}_0)$$

I derive

$$\frac{\frac{\partial}{\sum_{\ell \in \mathcal{K}\setminus\{0\}} \mu'_\ell(z_\ell)\partial z_\ell} \frac{\partial^{|\mathcal{K}_0|-2}}{\prod_{\ell \in \mathcal{K}_0^*} \partial z_\ell} PY_k(z)}{\frac{\partial}{\sum_{\ell \in \mathcal{K}\setminus\{0\}} \mu'_\ell(z_\ell)\partial z_\ell} \frac{\partial^{|\mathcal{K}_0|-2}}{\prod_{\ell \in \mathcal{K}_0^*} \partial z_\ell} P_k(z)} = \mathbf{E}[\mathrm{MTR}_k(U_i) | U_i \in \mathcal{U}(\mathcal{K}_0)] \tag{A12}$$

When $|\mathcal{K}_0| = 2$ and $K = 2$, Equation A12 is implicitly derived on Page 2610 of Mountjoy (2022). □

A.4 Complier group probabilities for C_{xya} and C_{xyb}

Proof. I derive Equation 6 (for $\mathbf{P}[U_i \in C_{xya}]$), and note that Equation 7 holds symmetrically.

First, I apply the definition of C_{xya} in terms of inequalities that the latent utility u must satisfy, which I simplify by eliminating redundant inequalities.

$$\begin{aligned} \mathbf{P}[U_i \in C_{xya}] &= \mathbf{P}[U_{ia} \in [x, x+1], U_{ib} \in [y, y+1], U_{ia} - x \geq U_{ib} - y] \\ &= \mathbf{P}[U_{ia} \leq x+1, U_{ib} \geq y, U_{ia} - x \geq U_{ib} - y] \end{aligned}$$

Next, I use E_1, E_2, and E_3 to denote the events $U_{ia} \leq x+1$, $U_{ib} \geq y$, and $U_{ia} - x \geq U_{ib} - y$, respectively. I note that $\mathbf{P}[\neg E_1 \wedge \neg E_2 \wedge \neg E_3] = 0$, as if U_{ib} were smaller than y (the event $\neg E_1$), and U_{ia} were larger than $x+1$ (the event $\neg E_2$), then $U_{ia} - x > 1 > 0 > U_{ib} - y$ (the event E_3). I apply this, and then apply elementary logical operations, to derive

$$\begin{aligned}
\mathbf{P}[U_i \in C_{xya}] &= \mathbf{P}[E_1 \wedge E_2 \wedge E_3] \\
&= \mathbf{P}[(E_1 \wedge E_2 \wedge E_3) \vee (\neg E_1 \wedge \neg E_2 \wedge \neg E_3)] \\
&= \mathbf{P}[(E_1 \vee \neg E_3) \wedge (E_2 \vee \neg E_1) \wedge (E_3 \vee \neg E_2)] \\
&= \mathbf{P}[\neg(\neg E_1 \wedge E_3) \wedge \neg(\neg E_2 \wedge E_1) \wedge \neg(\neg E_3 \wedge E_2)] \\
&= 1 - \mathbf{P}[(\neg E_1 \wedge E_3) \vee (\neg E_2 \wedge E_1) \vee (\neg E_3 \wedge E_2)] \\
&= 1 - \mathbf{P}[\neg E_1 \wedge E_3] - \mathbf{P}[\neg E_2 \wedge E_1] - \mathbf{P}[\neg E_3 \wedge E_2]
\end{aligned}$$

where the last step relies on the mutual exclusiveness of the three events $\neg E_1 \wedge E_3$, $\neg E_2 \wedge E_1$, and $\neg E_3 \wedge E_2$.

Lastly, I substitute

$$\begin{aligned}
\mathbf{P}[\neg E_1 \wedge E_3] &= \mathbf{P}[U_{ia} \geq x+1 \wedge U_{ia} - (x+1) \geq U_{ib} - (y+1)] &&= P_a(x+1, y+1) \\
\mathbf{P}[\neg E_2 \wedge E_1] &= \mathbf{P}[U_{ib} \leq y \wedge U_{ia} \leq x+1] &&= P_0(x+1, y) \\
\mathbf{P}[\neg E_3 \wedge E_2] &= \mathbf{P}[U_{ia} - x \leq U_{ib} - y \wedge U_{ib} \geq y] &&= P_b(x, y)
\end{aligned}$$

which yields Equation 6. □

A.5 Complier group mean potential outcome under a

Proof. I begin by simplying Equation A3 in the case $K = 2$, that is $P_a(z)$ expressed as an integral over the density of latent utilities $f(u)$. Similarly, I express $PY_a(z)$ as an integral over the density of latent utilities times the marginal treatment response, $\mathrm{MTR}_a(u) f(u)$.

$$P_a(z) = \int_{z_a}^{\infty} \int_{-\infty}^{u_a - z_a + z_b} f(u) du_b du_a \tag{A13}$$

$$PY_a(z) = \int_{z_a}^{\infty} \int_{-\infty}^{u_a - z_a + z_b} \mathrm{MTR}_a(u) f(u) du_b du_a \tag{A14}$$

I substitute Equations A13 and A14 into the right hand side of Equation 9, yielding

$$\begin{aligned}
&\frac{(PY_a(x,y+1)-PY_a(x+1,y+2))-(PY_a(x,y)-PY_a(x+1,y+1))}{(P_a(x,y+1)-P_a(x+1,y+2))-(P_a(x,y)-P_a(x+1,y+1))}\\
&=\frac{\int_x^{x+1}\int_{u_a-x+y}^{u_a-x+y+1}\mathrm{MTR}_a(u)f(u)du_b du_a}{\int_x^{x+1}\int_{u_a-x+y}^{u_a-x+y+1}f(u)du_b du_a}\\
&=\frac{\int_{C_{xyb}\cup C_{x,y+1,a}}\mathrm{MTR}(a)(u)f(u)du}{\int_{C_{xyb}\cup C_{x,y+1,a}}f(u)du}
\end{aligned}\tag{A15}$$

where the last equality in Equation A15 applies the substitutions

$$\begin{aligned}
C_{xyb}&=\{u:u_a\in[x,x+1],u_b\in[y,y+1],u_a-x\le u_b-y\}\\
&=\{u:u_a\in[x,x+1],u_b\in[u_a-x+y,y+1]\}
\end{aligned}$$

and

$$\begin{aligned}
C_{x,y+1,a}&=\{u:u_a\in[x,x+1],u_b\in[y+1,y+2],u_a-x\ge u_b-(y+1)\}\\
&=\{u:u_a\in[x,x+1],u_b\in[y+1,u_a-x+y+1]\}
\end{aligned}$$

The last line of Equation A15 is exactly $\mathbf{E}[\mathrm{MTR}_a(U_i)|U_i\in C_{xyb}\cup C_{x,y+1,a}]=\mathbf{E}[Y_{ia}|U_i\in C_{xyb}\cup C_{x,y+1,a}]$, which yields Equation 9. $\square$

A.6 Proposition 3

Below, and in Appendix A.7 and A.8, I use the "little o notation" $g(x)=o(h(x))$, for functions $g(x)$ and $h(x)$ such that $\lim_{x\to 0}\frac{g(x)}{|h(x)|}=0$. I repeatedly apply two well known properties of $o(\cdot)$: $ao(h(x))+bo(h(x))=o(h(x))$ for any constants $a>0$ and $b>0$, and $h_2(x)o(h_1(x))=o(h_1(x)h_2(x))$.

Proof. Below, I establish that the Manski (1990) lower bound on $\mathbf{E}[Y_{i0}|U_i\in C_{01a}(\epsilon)]$ is $\underline{Y}$ as $\epsilon\to 0$. By a symmetric argument, the lower bound on $\mathbf{E}[Y_{i0}|U_i\in C_{10b}(\epsilon)]$ is $\underline{Y}$ as $\epsilon\to 0$. These lower bounds are independently achievable, as they are tied to the unobserved $\mathbf{E}[Y_{i0}|U_i\in C_{01b}(\epsilon)]$ and $\mathbf{E}[Y_{i0}|U_i\in C_{10a}(\epsilon)]$, respectively, both taking on their maximum possible value. As a result, the lower bound on $\mathbf{E}[Y_{i0}|U_i\in C_{01a}(\epsilon)\cup C_{10b}(\epsilon)]$ is also $\underline{Y}$ as $\epsilon\to 0$.

I proceed in three steps. First, I derive the Manski (1990) lower bound on $\mathbf{E}[Y_{i0}|U_i \in C_{01a}(\epsilon)]$ as a function of identified mean potential outcomes and complier group probabilities. Second, I derive the limits of these identified mean potential outcomes and complier group probabilities as $\epsilon \to 0$. Third, I apply these expressions to take the limit of the lower bound on $\mathbf{E}[Y_{i0}|U_i \in C_{01a}(\epsilon)]$ as $\epsilon \to 0$, and show it approaches $\underline{Y}$; by a symmetric argument, the upper bound approaches $\overline{Y}$.

First, note that $\mathbf{E}[Y_{i0}|U_i \in C_{01a}(\epsilon)]$ must satisfy

$$\mathbf{E}[Y_{i0}|U_i \in C_{01a}(\epsilon)] \geq \max\left\{\underline{Y}, \frac{\mathbf{E}[Y_{i0}|U_i \in C_{01a}(\epsilon) \cup C_{01b}(\epsilon)] - \frac{\mathbf{P}[U_i \in C_{01b}(\epsilon)]}{\mathbf{P}[U_i \in C_{01a}(\epsilon) \cup C_{01b}(\epsilon)]}\overline{Y}}{\frac{\mathbf{P}[U_i \in C_{01a}(\epsilon)]}{\mathbf{P}[U_i \in C_{01a}(\epsilon) \cup C_{01b}(\epsilon)]}}\right\} \tag{A16}$$

Equation A16 combines two inequalities. First, $\mathbf{E}[Y_{i0}|U_i \in C_{01a}(\epsilon)]$ must satisfy the assumed lower bound ($\underline{Y}$) in Assumption Y.2. Second, $\mathbf{E}[Y_{i0}|U_i \in C_{01a}(\epsilon)]$ must be larger than the smallest possible value consistent with the identified potential outcome mean $\mathbf{E}[Y_{i0}|U_i \in C_{01a}(\epsilon) \cup C_{01b}(\epsilon)]$, which is binding when $\mathbf{E}[Y_{i0}|U_i \in C_{01b}(\epsilon)]$ takes on the assumed upper bound ($\overline{Y}$) in Assumption Y.2.

Second, I apply the approximations

$$f(u) = f(0) + o(1)$$

$$\text{MTR}_0(u)f(u) = \text{MTR}_0(0)f(0) + o(1)$$

which are valid by continuity of $f(u) > 0$ (Assumption TRUM.2), and of $\text{MTR}_0(u)$ (Assumption Y.1), respectively. I apply these approximations to $\mathbf{P}[U_i \in C_{01a}(\epsilon)]$, yielding

$$\begin{aligned}
\mathbf{P}[U_i \in C_{01a}(\epsilon)] &= \int_0^{\epsilon}\int_{\epsilon}^{u_a+\epsilon} f(u)du \\
&= \int_0^{\epsilon}\int_{\epsilon}^{u_a+\epsilon} \left(f(0) + o(1)\right) du \\
&= \frac{\epsilon^2}{2} f(0) + o(\epsilon^2)
\end{aligned}$$

Identical steps imply

$$\begin{aligned}\mathbf{P}[U_i \in C_{01b}(\epsilon)] &= \frac{\epsilon^2}{2} f(0) + o(\epsilon^2) \\ \mathbf{P}[U_i \in C_{01a}(\epsilon) \cup C_{01b}(\epsilon)] &= \epsilon^2 f(0) + o(\epsilon^2) \\ \mathbf{P}[U_i \in C_{01a}(\epsilon) \cup C_{01b}(\epsilon)]\mathbf{E}[Y_{i0}|U_i \in C_{01a}(\epsilon) \cup C_{01b}(\epsilon)] &= \epsilon^2 \mathrm{MTR}_0(0) f(0) + o(\epsilon^2)\end{aligned}$$

Third, substituting these expressions, applying $o(\epsilon^2)/\epsilon^2 = o(1)$, and substituting $\mathrm{MTR}_0(0) = \frac{1}{2}(\underline{Y} + \overline{Y})$ (as assumed in Proposition 3) into Equation A16 yields

$$\begin{aligned}\mathbf{E}[Y_{i0}|U_i \in C_{01a}(\epsilon)] &\geq \max\left\{\underline{Y}, \frac{\epsilon^2 \mathrm{MTR}_0(0) f(0) + o(\epsilon^2) - \left(\frac{\epsilon^2}{2} f(0) + o(\epsilon^2)\right)\overline{Y}}{\frac{\epsilon^2}{2} f(0) + o(\epsilon^2)}\right\} \\ &= \max\left\{\underline{Y}, \frac{2\mathrm{MTR}_0(0) + o(1) - \overline{Y}}{1 + o(1)}\right\} \\ &= \max\left\{\underline{Y}, \frac{\underline{Y} + o(1)}{1 + o(1)}\right\}\end{aligned}$$

which approaches $\underline{Y}$ as $\epsilon \to 0$. □

A.7 Proposition 4

Proof. First, note that Assumption MLATR is equivalent to, for $d \in \{0, a, b\}$,

$$\begin{aligned}&\mathbf{E}[Y_{id}|U_i \in C_{xyk}(\epsilon)] - \mathbf{E}[Y_{id}|U_i \in C_{x+w_a,y+w_b,k}(\epsilon)] \geq 0 \\ &\qquad \Leftrightarrow \mathbf{E}[Y_{id}|U_i \in C_{x'y'k}(\epsilon)] - \mathbf{E}[Y_{id}|U_i \in C_{x'+w_a,y'+w_b,k}(\epsilon)] \geq 0\end{aligned}$$

Further note that

$$\begin{aligned}&\mathbf{E}[Y_{id}|U_i \in C_{xyk}(\epsilon)] - \mathbf{E}[Y_{id}|U_i \in C_{x+w_a,y+w_b,k}(\epsilon)] \geq 0 \\ &\Leftrightarrow \mathbf{E}[Y_{id}|U_i \in C_{xyk(\epsilon)}]\frac{\mathbf{P}[U_i \in C_{x+w_a,y+w_b,k}(\epsilon)] - \mathbf{P}[U_i \in C_{xyk}(\epsilon)]}{\epsilon^3} \\ &\quad + \frac{\mathbf{E}[Y_{id}|U_i \in C_{xyk}(\epsilon)]\mathbf{P}[U_i \in C_{xyk}(\epsilon)] - \mathbf{E}[Y_{id}|U_i \in C_{x+w_a,y+w_b,k}(\epsilon)]\mathbf{P}[U_i \in C_{x+w_a,y+w_b,k}(\epsilon)]}{\epsilon^3} \geq 0\end{aligned}$$

Taking limits of the second inequality yields, in the limit as $\epsilon \to 0$

$$\mathbf{E}[Y_{id}|U_i \in C_{xyk}(\epsilon)] - \mathbf{E}[Y_{id}|U_i \in C_{x+w_a,y+w_b,k}(\epsilon)] \geq 0$$
$$\Leftrightarrow -\frac{1}{2}f(0)\left(w_a\frac{\partial \text{MTR}_d(0)}{\partial u_a} + w_b\frac{\partial \text{MTR}_d(0)}{\partial u_b}\right) \geq 0$$

excluding the case where $w_a\frac{\partial \text{MTR}_d(0)}{\partial u_a} + w_b\frac{\partial \text{MTR}_d(0)}{\partial u_b} = 0$.

The same limiting equivalence holds for (x', y') in place of (x, y), and therefore, in the limit as $\epsilon \to 0$, Assumption MLATR holds. □

A.8 Proposition 6

Proof. I proceed in two steps. First, I show that $\mathbf{E}[Y_{i0}|U_i \in C_{01a}(\epsilon) \cup C_{01b}(\epsilon)]$ approaches $\text{MTR}_0(0)$ as $\epsilon \to 0$; an identical argument applies to $\mathbf{E}[Y_{i0}|U_i \in C_{10a}(\epsilon) \cup C_{10b}(\epsilon)]$. Second, I show that $\frac{\mathbf{P}[U_i \in C_{01a}(\epsilon)]}{\mathbf{P}[U_i \in C_{01a}(\epsilon) \cup C_{10b}(\epsilon)]}$ approaches $\frac{1}{2}$ as $\epsilon \to 0$; an identical argument applies to $\frac{\mathbf{P}[U_i \in C_{01a}(\epsilon)]}{\mathbf{P}[U_i \in C_{01a}(\epsilon) \cup C_{01b}(\epsilon)]}$ and $\frac{\mathbf{P}[U_i \in C_{10b}(\epsilon)]}{\mathbf{P}[U_i \in C_{10a}(\epsilon) \cup C_{10b}(\epsilon)]}$. It immediately follows that both bounds in Equation 11 approach $\text{MTR}_0(0)$ as $\epsilon \to 0$. An identical argument applies to Equations 12 and 13, and Proposition 6 holds.

First, note that

$$\min_{u \in C_{01a}(\epsilon) \cup C_{01b}(\epsilon)} \text{MTR}_0(u) \leq \mathbf{E}[Y_{i0}|U_i \in C_{01a}(\epsilon) \cup C_{01b}(\epsilon)] \leq \max_{u \in C_{01a}(\epsilon) \cup C_{01b}(\epsilon)} \text{MTR}_0(u)$$

By continuity of $\text{MTR}_0(u)$ (Assumption Y.1),

$$\lim_{\epsilon \to 0} \min_{u \in C_{01a}(\epsilon) \cup C_{01b}(\epsilon)} \text{MTR}_0(u) = \lim_{\epsilon \to 0} \max_{u \in C_{01a}(\epsilon) \cup C_{01b}(\epsilon)} \text{MTR}_0(u) = \text{MTR}_0(0)$$

Therefore

$$\lim_{\epsilon \to 0} \mathbf{E}[Y_{i0}|U_i \in C_{01a}(\epsilon) \cup C_{01b}(\epsilon)] = \text{MTR}_0(0)$$

Second, by continuity of $f(u)$ (Assumption TRUM.2), and from Section A.6,

$$\mathbf{P}[U_i \in C_{01a}(\epsilon)] = \frac{\epsilon^2}{2}f(0) + o(\epsilon^2)$$

Therefore

$$\frac{\mathbf{P}[U_i \in C_{01a}(\epsilon)]}{\mathbf{P}[U_i \in C_{01a}(\epsilon) \cup C_{10b}(\epsilon)]} = \frac{\frac{\epsilon^2}{2}f(0) + o(\epsilon^2)}{\epsilon^2 f(0) + o(\epsilon^2)} = \frac{1}{2}\left(\frac{1 + o(1)}{1 + o(1)}\right)$$

Therefore

$$\lim_{\epsilon \to 0} \frac{\mathbf{P}[U_i \in C_{01a}]}{\mathbf{P}[U_i \in C_{01a} \cup C_{10b}]} = \frac{1}{2}$$

$\square$