Modeling selection into unordered treatments: An equivalence result*

John Loeser†

October 18, 2023

Abstract

When individuals self-select into multiple unordered treatments, additional assumptions are needed to identify treatment effects with instrumental variables. Suppose each treatment is targeted by a single instrument, then the following assumptions enabling identification of treatment effects are effectively equivalent: no defiers, identical compliers, and an additive random utility model of treatment selection. This equivalence extends approaches to identification and falsification derived under each assumption, and suggests new ones: no defiers is equivalent to a set of testable restrictions on choice probabilities, and all treatment effects are identified for individuals indifferent between control and all treatments.

*`jloeser@worldbank.org`. This research benefited from suggestions and comments from Jack Mountjoy, Matthew Pecenco, and Yotam Shem-Tov. The views expressed do not reflect the views of the World Bank. All errors are my own.

†Development Impact Evaluation, World Bank

When individuals self-select into treatments, under what conditions are treatment effects identified? With a single treatment, a weighted average of treatment effects is identified under an economically interpretable "monotonicity" assumption on an instrumental variable (Imbens & Angrist, 1994; Heckman & Vytlacil, 2005): increases in the instrument induce some individuals to shift from treatment to control, but do not induce any to shift from control to treatment. In contrast, with multiple treatments, the widely applied result with a single treatment that two-stage least squares estimates a local average treatment effect no longer holds, even when there is at least one instrument per treatment — additional assumptions are then necessary to identify treatment effects (Behaghel et al., 2013; Kirkeboen et al., 2016).

A growing set of assumptions have been proposed that enable identification of treatment effects when individuals self-select into multiple treatments; while each set of assumptions yields distinct approaches to identification of treatment effects, each collapses to monotonicity when treatment is binary. A key source of this multiplicity is that monotonicity encapsulates two distinct classes of assumptions that are equivalent when treatment is binary, but must be combined with multiple treatments. The first are targeting (Lee & Salanié, 2023) or partial monotonicity (Mogstad et al., 2021; Mountjoy, 2022) assumptions, that restrict the sign of responses to changes in the value of the instruments. The second are homogeneity assumptions, that restrict heterogeneity across compliers to changes in the value of the instrument; these include no defiers (Navjeevan & Pinto, 2022), identical compliers (Mountjoy, 2022), and an additive random utility model of treatment selection (Lee & Salanié, 2018).

In this paper I establish an equivalence result, clarifying links across assumptions enabling identification of treatment effects with multiple unordered treatments: when each treatment is targeted by a single instrument, these three homogeneity assumptions are effectively equivalent. This equivalence result immediately enables results or intuitions derived under one assumption to be applied under others.

In Section 1, I formalize these assumptions on selection of individuals into multiple treatments. As in Mountjoy (2022), I maintain that responses to changes in the value of the instruments satisfy unordered partial monotonicity ("UPM"), that each instrument shifts agents towards the instrument's associated choices and away from other choices. I show this assumption is equivalent to assuming that individuals make utility maximizing choices when selecting into treatment, and each instrument increases individual utilities associated with its targeted choice, as if a price or characteristic of the targeted choice.

I then consider three additional assumptions that restrict the heterogeneity of treatment responses to changes in the value of the instruments across individuals. Existing work has developed approaches to identification of treatment effects and falsification tests under each assumption.

- No defiers ("ND") imposes a global notion of monotonicity: for any change in the value of the instruments, no two individuals have opposing treatment responses.
- Identical compliers ("IC") imposes a marginal notion of monotonicity: the same individuals are marginal between a pair of treatments, whether revealed by small changes in either instrument targeting each treatment.[1]
- The Additive Random Utility Model ("ARUM") imposes that agent utility is additively separable in unobserved heterogeneity and the instrument.

In Section 2, I show that conditional on UPM, and additional technical assumptions, ND, IC, and ARUM are equivalent. This equivalence extends a result from Vytlacil (2002), that ND and ARUM are equivalent when treatment is binary, to the setting with multiple unordered treatments. The proof builds closely on Mogstad et al. (2021), who show that with multiple instruments and binary treatment, ND is effectively equivalent to homogeneous instrument sensitivity. Under additional technical assumptions, homogeneous instrument sensitivity is equivalent to ARUM. I apply their approach and extend it to IC — ND and IC are equivalent to homogeneous instrument sensitivity, and therefore ARUM.

In Section 3, I apply this equivalence to results on identification of treatment effects and falsification tests. Immediately, results under UPM and ARUM (Lee & Salanié, 2018; Allen & Rehbeck, 2019; Bhattacharya, 2023) and results under UPM and IC (Mountjoy, 2022) apply under UPM and either ND, IC, or ARUM.

I then leverage insights from this equivalence to derive a novel identification and falsifiability result. First, the average effect of any treatment relative to any other, among individuals indifferent between all treatments, is identified with local variation

[1] This assumption is closely related to the assumption of comparable compliers from Mountjoy (2022), which imposes the weaker restriction that individuals with the same average potential outcomes are marginal. I discuss the distinction between assumptions on selection into treatment, on which this paper focuses, and assumptions on outcomes, which are often sufficient for identification and need not imply assumptions on selection, in Section 1.2.

in the instruments. Second, conditional on UPM, ND is equivalent to a set of testable restrictions on choice probabilities.

1 Modeling selection into multivalued treatment

This section describes a model of selection into multivalued treatment and associated assumptions. Section 1.1 describes the selection framework with two treatments and two instruments. Section 1.2 describes and interprets potential additional assumptions related to monotonicity on the selection model that enable identification of treatment effects. Section 1.3 interprets the relationship between these and other assumptions on the selection model through their restrictions on admissible treatment response graphs. Section 1.4 develops additional technical assumptions.

1.1 Selection into multivalued treatment

Let $\mathcal{I}$ denote the population of individuals $i \in \mathcal{I}$, with an associated probability measure $\mathbf{P}$ and expectation $\mathbf{E}$. Let $D_i(z) \in \{0, 2, 4\}$ denote the potential treatment status of individual i if their instrument $Z_i \equiv (Z_{i2}, Z_{i4})$ were set to $z \equiv (z_2, z_4)$, where Z_i and z have support on a 2-dimensional interval $\mathcal{Z} \subseteq \mathbb{R}^2$. Further let $D_{id}(z) \equiv \mathbf{1}[D_i(z) = d]$ indicate that individual i would have treatment status d if their instrument were set to z.

This notation builds closely on Mountjoy (2022), and I use their context throughout this section for concreteness. In their empirical setting, the values of the treatment status 0, 2, and 4 correspond to not attending college, initially attending two-year college, and initially attending four-year college, respectively. In their application, the values of the instrument z_2 and z_4 correspond to distance to the nearest two-year college and distance to the nearest four-year college, respectively.

1.2 Monotonicity

With a binary treatment and scalar instrument, the assumption that treatment status is increasing in the value of the instrument ("monotonicity", in Imbens & Angrist, 1994) implies no defiers; in contrast, with multiple unordered treatments, common restrictions on the sign and heterogeneity of treatment responses are no longer nested. Section 1.2.1 describes an assumption that restricts the sign of treatment responses to changes in the value of each instrument, unordered partial monotonicity, and links it to a random utility model. Section 1.2.2 describes assumptions that restrict heterogeneity in responses across individuals: identical compliers, no defiers, and an additive random utility model.

1.2.1 Unordered partial monotonicity and targeting

In many applications, the instrument z acts similarly to (or is) the vector of prices of the treatments. When the price of a given treatment increases, the canonical random utility model restricts the potential treatment responses: marginal individuals shift out of the now more expensive treatment into other treatments, but no individuals are induced to shift between treatments that did not experience price changes. Mountjoy (2022) formalizes this assumption, which I restate with the notation in Section 1.1.

Assumption UPM (Unordered Partial Monotonicity)**.** *For all $i \in \mathcal{I}$, and $(z_2, z_4), (z_2', z_4) \in \mathcal{Z}$ with $z_2' < z_2$,*

$$D_{i2}(z_2', z_4) \geq D_{i2}(z_2, z_4),\ D_{i0}(z_2', z_4) \leq D_{i0}(z_2, z_4),\ D_{i4}(z_2', z_4) \leq D_{i4}(z_2, z_4)$$

For all $i \in \mathcal{I}$, and $(z_2, z_4), (z_2, z_4') \in \mathcal{Z}$ with $z_4' < z_4$,

$$D_{i4}(z_2, z_4') \geq D_{i4}(z_2, z_4),\ D_{i0}(z_2, z_4') \leq D_{i0}(z_2, z_4),\ D_{i2}(z_2, z_4') \leq D_{i2}(z_2, z_4)$$

In the context of college attendance choices in Mountjoy (2022), Assumption UPM imposes the following restrictions on treatment status:

- Decreasing the distance to the nearest two-year college causes individuals to shift from not attending college to two-year college, and to shift from attending four-year college to attending two-year college. It does not cause any individuals to stop attending two-year college, or to shift between not attending college and four-year college.
- Similarly, decreasing the distance to the nearest four-year college causes individuals to shift from not attending college to four-year college, and to shift from attending two-year college to attending four-year college. It does not cause any individuals to stop attending four-year college, or to shift between not attending college and two-year college.

These restrictions are motivated by individuals making their utility-maximizing choice of college attendance, with the utility from attending a college decreasing in the distance to the college. Through the lens of the random utility model, Assumption UPM imposes restrictions on how instruments (distances to the nearest two-year college and four-year college) can affect individuals' utilities, in the same manner as the "targeting" assumption proposed by Lee & Salanié (2023).

Assumption TRUM (Targeted Random Utility Model)**.** *For all $i \in \mathcal{I}$, treatment status $D_i(z)$ satisfies*

$$D_i(z) = \arg\max_{d \in \{0,2,4\}} V_{id}(z) \tag{1}$$

for all $z \in \mathcal{Z}$. Further, $V_{i0}(z) = 0$, $V_{i2}(z) = U_{i2} - \mu_{i2}(z_2)$, and $V_{i4}(z) = U_{i4} - \mu_{i4}(z_4)$, where μ_{i2} and μ_{i4} are increasing functions of z_2 and z_4, respectively.

Proposition 1. *Assumptions UPM and TRUM are equivalent.*

Proof. I establish equivalence by showing that Assumption TRUM implies Assumption UPM, and then that any deviation from Assumption TRUM implies a violation of Assumption UPM.

Assumption TRUM $\Rightarrow$ Assumption UPM Take any $(z_2, z_4), (z_2', z_4) \in \mathcal{Z}$ with $z_2' < z_2$. Let $z \equiv (z_2, z_4)$ and $z' \equiv (z_2', z_4)$. By Assumption TRUM, for all $i \in \mathcal{I}$, $V_{i0}(z) = V_{i0}(z')$, $V_{i2}(z) \leq V_{i2}(z')$, and $V_{i4}(z) = V_{i4}(z')$. As Equation 1 holds, for all $i \in \mathcal{I}$, $D_{i2}(z') \geq D_{i2}(z)$, $D_{i0}(z') \leq D_{i0}(z)$, and $D_{i4}(z') \leq D_{i4}(z)$.

Take any $(z_2, z_4), (z_2, z_4') \in \mathcal{Z}$ with $z_4' < z_4$. Let $z \equiv (z_2, z_4)$ and $z' \equiv (z_2, z_4')$. By Assumption TRUM, for all $i \in \mathcal{I}$, $V_{i0}(z) = V_{i0}(z')$, $V_{i2}(z) = V_{i2}(z')$, and $V_{i4}(z) \leq V_{i4}(z')$. As Equation 1 holds, for all $i \in \mathcal{I}$, $D_{i4}(z') \geq D_{i4}(z)$, $D_{i0}(z') \leq D_{i0}(z)$, and $D_{i2}(z') \leq D_{i2}(z)$.

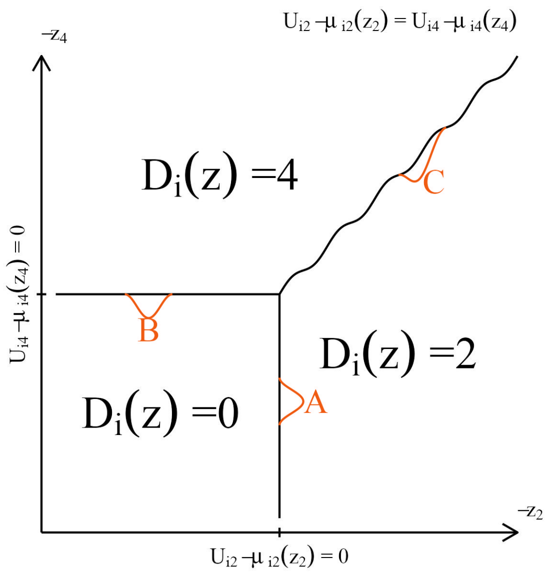
Assumption UPM therefore holds.

¬Assumption TRUM $\Rightarrow$ ¬Assumption UPM Assumption TRUM implies a system of inequalities characterize treatment choice:

$$\begin{aligned}
&D_i(z) = 0 \Leftrightarrow U_{i2} - \mu_{i2}(z_2) \leq 0 \;\&\; U_{i4} - \mu_{i4}(z_4) \leq 0 \\
&D_i(z) = 2 \Leftrightarrow U_{i2} - \mu_{i2}(z_2) \geq 0 \;\&\; U_{i2} - \mu_{i2}(z_2) - U_{i4} - \mu_{i4}(z_4) \geq 0 \\
&D_i(z) = 4 \Leftrightarrow U_{i4} - \mu_{i4}(z_4) \geq 0 \;\&\; U_{i2} - \mu_{i2}(z_2) - U_{i4} - \mu_{i4}(z_4) \leq 0
\end{aligned}$$

An example solution to this system of inequalities is plotted in Figure 1; consider the possible deviations from Assumption TRUM, labeled A, B, and C.[2]

Figure 1: Unordered partial monotonicity implies a targeted random utility model



Deviation A implies there exists z_2, z_4, z_4' with $z_4' < z_4$ such that $D_i(z_2, z_4) = 2$ and $D_i(z_2, z_4') = 0$, contradicting Assumption UPM.

Deviation B implies there exists z_2, z_2', z_4 with $z_2' < z_2$ such that $D_i(z_2, z_4) = 0$ and $D_i(z_2', z_4) = 4$, contradicting Assumption UPM.

Deviation C implies there exists z_2, z_2', z_4 with $z_2' < z_2$ such that $D_i(z_2, z_4) = 2$ and $D_i(z_2', z_4) = 4$, contradicting Assumption UPM. $\square$

1.2.2 Identical compliers, no defiers, and additive random utility

Conditional on Assumption UPM, which restricts the sign of treatment responses to changes in the value of the instrument, I consider three assumptions that restrict the heterogeneity of treatment responses.

Assumption IC (Identical Compliers)**.** *For all* $(z_2, z_4) \in \mathcal{Z}$, $I \subseteq \mathcal{I}$

$$\lim_{z_2' \uparrow z_2} \mathbf{P}\left[i \in I \middle| D_i(z_2', z_4) = 2, D_i(z_2, z_4) = 4\right] = \lim_{z_4' \downarrow z_4} \mathbf{P}\left[i \in I \middle| D_i(z_2, z_4') = 2, D_i(z_2, z_4) = 4\right]$$

[2] I present an alternative non-graphical proof that Assumption UPM implies Assumption TRUM in Appendix.

Assumption IC imposes "marginal monotonicity": the same individuals are marginal between treatment statuses 2 and 4, whether revealed by small changes in z_2 or z_4. Assumption IC is closely related to "comparable compliers" from Mountjoy (2022); rather than assuming that the average potential outcomes of individuals induced to shift from treatment status 2 to treatment status 4 by small increases in z_2 and small decreases in z_4 are the same, I assume that the average types of these individuals are the same. Comparable types is a strictly stronger assumption, as, for example, comparable potential outcomes is satisfied whenever selection is independent of potential outcomes. I focus on comparable types to separate assumptions on selection from assumptions on potential outcomes conditional on $D_i(\cdot)$.

Assumption ND (No Defiers)**.** *For all $z, z' \in \mathcal{Z}$, and $(d, d') \in \{0, 2, 4\}$ with $d \neq d'$*

$$\mathbf{P}\left[D_i(z) = d, D_i(z') = d'\right] = 0 \text{ or } \mathbf{P}\left[D_i(z) = d', D_i(z') = d\right] = 0$$

Assumption ND imposes "global monotonicity": for any change in the value of the instruments, no two individuals have opposing treatment responses. Navjeevan & Pinto (2022) discuss identification under Assumption ND with discrete instruments.

Assumption ARUM (Additive Random Utility Model)**.** *For all $i \in \mathcal{I}$ and $z \in \mathcal{Z}$, treatment status satisfies Equation 1 where $V_{i0}(z) = 0$, $V_{i2}(z) = U_{i2} + \mu_2(z)$, $V_{i4}(z) = U_{i4} + \mu_4(z)$.*

Assumption ARUM imposes strong restrictions on heterogeneity of treatment responses: for any change in the value of the instruments, all individuals agree on the changes in relative utility across treatments. Lee & Salanié (2018) discuss identification of treatment effects under Assumption ARUM.

Assumptions IC, ND, and ARUM are redundant conditional on Assumption UPM for a binary treatment and instrument. In this case, Assumption UPM corresponds to the monotonicity assumption in Imbens & Angrist (1994), and increases in the binary instrument must shift agents away from control and into treatment. Without multivalued treatment, Assumption IC holds vacuously, Assumption ND is implied by monotonicity, and Vytlacil (2002) showed that Assumption ARUM is implied by monotonicity. However, with multiple treatments and multiple instruments, each of these assumptions puts additional restrictions on choices conditional on Assumption UPM.

1.3 Relationships across assumptions on selection

Assumption ND can be equivalently stated in terms of the permissible directed graphs of flows between treatments in response to a change in the value of the instrument. Define the "treatment response graph" $G^{(z,z')}$ for a change in the value of the instrument from z to z', with element $(d,d') \in \{0,2,4\}^2$ equal to $G^{(z,z')}_{(d,d')}$ by

$$G^{(z,z')}_{(d,d')} \equiv \mathbf{1}\{d \neq d'\}\mathbf{1}\{\mathbf{P}[D_i(z) = d, D_i(z') = d'] > 0\} \tag{2}$$

The element (d,d') of the treatment response graph $G^{(z,z')}$ is 1 if the change of the instrument $z \to z'$ induces individuals to shift $d \to d'$, and 0 otherwise.[3]

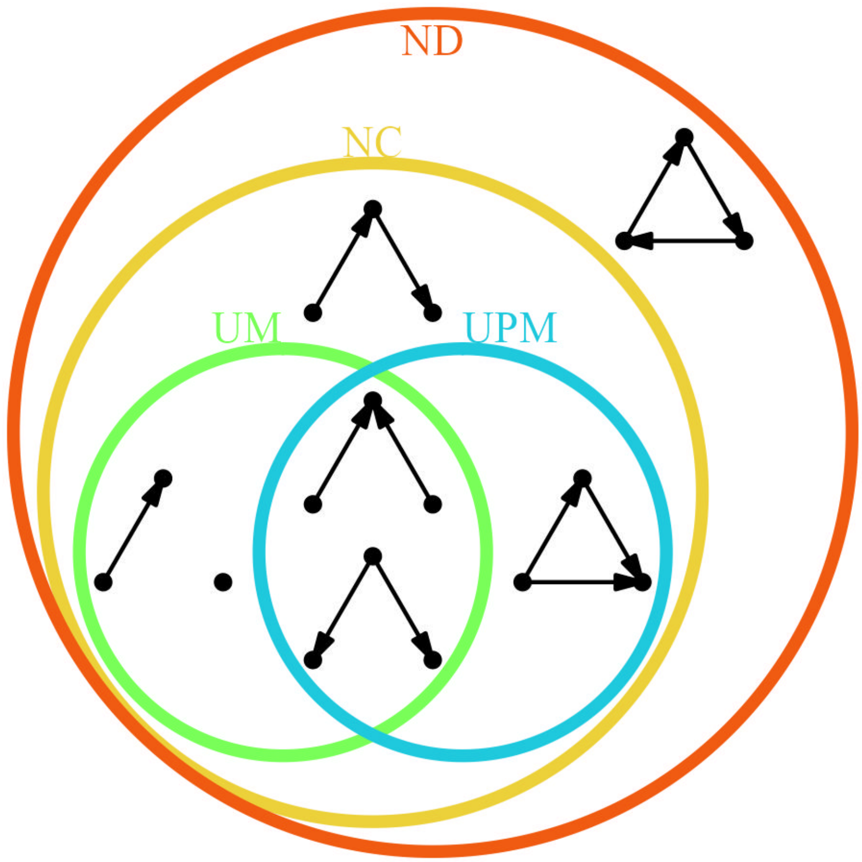
Assumption ND can be equivalently stated using the treatment response graph as follows: for any $d,d' \in \{0,2,4\}$, either $G^{(z,z')}_{(d,d')} = 0$ or $G^{(z,z')}_{(d',d)} = 0$. That is, there is no change in the value of the instrument from z to z' that both induces individuals to shift $d \to d'$ and also $d' \to d$.

Figure 2 plots the 6 unique treatment response graphs (up to permutations of nodes) that are consistent with no defiers with 3 treatment values. Conditional on Assumption ND, additional assumptions place additional restrictions on the set of permissible treatment response graphs. I discuss the restrictions placed by 3 additional assumptions: no cycles, unordered monotonicity (Heckman & Pinto, 2018), and Assumption UPM.

First, "no cycles" ("NC") imposes that there are no cycles in the treatment response graph $G^{(z,z')}$. This is a stronger assumption than Assumption ND – it rules out the case, for example, where a change in the value of the instruments induces the treatment flows $\{0 \to 2, 2 \to 4, 4 \to 0\}$. Interpreting flows of individuals across treatments as a revealed preference, a cycle implies a form of preference heterogeneity: the change in the value of the instruments must have strictly increased the relative value of 2 to 0 for individuals shifting $0 \to 2$, of 4 to 2 for agents shifting $2 \to 4$, and of 0 to 4 for agents shifting $4 \to 0$. These changes in relative values are not simultaneously possible for a single agent. Cycles challenge identification of treatment

[3]The admissible treatment response graphs discussed here are closely related to admissible binary response matrices under unordered monotonicity in Heckman & Pinto (2018) and under minimal monotonicity (equivalent to no defiers) in Navjeevan & Pinto (2022); binary response matrices carry additional information on restrictions to responses across multiple values of an instrument, which Heckman & Pinto (2018) and Navjeevan & Pinto (2022) use to establish results on identification with discrete variation in instruments.

Figure 2: Feasible treatment response graphs conditional on no defiers



Notes: The set of unique feasible treatment response graphs $G^{(z,z')}$, defined in Equation 2, up to permutations of nodes possible under Assumption ND are presented in this figure. The subset of these graphs that are feasible when either no cycles, unordered monotonicity, or Assumption UPM is also imposed are included in the associated labeled circle of the assumption.

effects in a manner similar to defiance: a cycle implies that unobserved flows of individuals across treatments are possible without any corresponding change in observed treatment probabilities.

Second, "unordered monotonicity" ("UM"), analyzed by Heckman & Pinto (2018), imposes that there is no treatment that experiences both individuals shifting into that treatment and individuals shifting out of that treatment in response to a change in the value of the instruments.[4] That is, for all $d \in \{0, 2, 4\}$, either $\sum_{d' \in \{0,2,4\}} G^{(z,z')}_{(d,d')} = 0$ or

[4] Other work has analyzed assumptions that imply unordered monotonicity (Behaghel et al., 2013; Bhuller & Sigstad, 2023), and are therefore distinct from unordered partial monotonicity.

$\sum_{d' \in \{0,2,4\}} G^{(z,z')}_{(d',d)} = 0$. UM strengthens no cycles, by effectively imposing that there are two tiers of treatments for any change in the instruments: those that receive flows of individuals from other treatments, and those that send flows of individuals to other treatments. It therefore rules out two sets of treatment response graphs consistent with no cycles which imply three tiers of treatments: $\{0 \to 2, 2 \to 4\}$, and $\{0 \to 2, 2 \to 4, 0 \to 4\}$.

Third, Assumption UPM, analyzed by Mountjoy (2022), imposes that each instrument induces positive flows into its associated treatment and out of other treatments, but not between other treatments.[5] Assumption UPM implies, but is not implied by, additional restrictions on the set of permissible treatment response graphs. Consider as an example reducing both z_2 and z_4. Reducing z_2 must induce agents to shift $0 \to 2$ and $4 \to 2$, while reducing z_4 must induce agents to shift $2 \to 4$ and $0 \to 4$; reducing z_2 and then z_4 must shift agents from $0 \to 4$, while reducing z_4 and then z_2 must shift agents from $0 \to 2$. When combined with no defiers, only three sets of flows are therefore possible from reducing both z_2 and z_4: $\{0 \to 2, 0 \to 4\}$, $\{0 \to 2, 0 \to 4, 2 \to 4\}$, and $\{0 \to 2, 0 \to 4, 4 \to 2\}$. Applying the above reasoning to each possible change in the value of the instrument, Assumption ND and Assumption UPM jointly strengthen no cycles by implying transitivity of treatment flows: if a change in the value of the instruments shifts individuals $2 \to 4$ (i.e., strict revealed preference for 4 over 2), and not from $2 \to 0$ (i.e., weak revealed preference for 2 over 0), then it must also shift individuals from $0 \to 4$ (i.e., strict revealed preference for 4 over 0).

That Assumptions UPM and ND jointly imply transitivity of treatment flows suggests an equivalence with Assumption ARUM: both transitivity and the "increasing differences" property of the additive random utility model (Lee & Salanié, 2023) are equivalent to the statement that any change in the value of the instrument $z \to z'$ induces a weak ordering of treatments through the associated treatment response graph $G^{(z,z')}$.

[5]While Assumption UPM only imposes that each instrument induces *weakly* positive flows into its associated treatment and out of other treatments, the version of Assumption UPM analyzed by Mountjoy (2022), or Assumption UPM coupled with Assumptions TRUM.1, TRUM.2, and TRUM.3, imposes *strictly* positive flows. In the analysis in this section, I implicitly assume strictly positive flows.

1.4 Technical assumptions on selection

I consider the following additional technical assumptions on the targeted random utility model in Assumption TRUM (equivalent to Assumption UPM).

Assumption TRUM.1. *$\mu_{i2}(z_2)$ and $\mu_{i4}(z_4)$ are continuously differentiable functions of z for all $i \in \mathcal{I}$, $(z_2, z_4) \in \mathcal{Z}$.*

Assumption TRUM.2. *(U_{i2}, U_{i4}) are continuously distributed with strictly positive density on $\mathbb{R}^2$ conditional on (μ_{i2}, μ_{i4}).*

Assumption TRUM.3. *$\mu'_{i2}(z_2) > 0$ and $\mu'_{i4}(z_4) > 0$ for all $i \in \mathcal{I}$, $(z_2, z_4) \in \mathcal{Z}$.*

I impose three additional technical restrictions on the targeted random utility model. Assumption TRUM.1 imposes that utility is continuously differentiable in the instrument. Assumption TRUM.2 ensures that unobserved heterogeneity is never one-dimensional among marginal individuals; that is, for each "sensitivity" to the instrument (μ_{i2}, μ_{i4}) and each value of the instrument z, there are positive densities of individuals (U_{i2}, U_{i4}) who are indifferent between treatment statuses 0 and 2, between 0 and 4, between 2 and 4, and between 0, 2, and 4. Assumption TRUM.3 ensures that all individuals are responsive to both instruments.

2 Equivalence across assumptions on selection

In this section, I show that Assumptions IC, ND, and ARUM are equivalent conditional on Assumption UPM and Assumptions TRUM.1, TRUM.2, and TRUM.3. In Section 2.1, I consider a selection model that deviates from Assumption ARUM by introducing heterogeneous instrument sensitivity, and show it violates Assumptions IC and ND. In Section 2.2, I show that Assumptions IC, ND, and ARUM are all equivalent to homogeneous instrument sensitivity conditional on Assumption UPM and Assumptions TRUM.1, TRUM.2, and TRUM.3.

2.1 An example with heterogeneous instrument sensitivity

Consider the following generalization (suggested by Mountjoy, 2022) of the additive random utility model in Assumption ARUM that allows for heterogeneous instrument sensitivity, while satisfying Assumption UPM.

$$V_{i0}(z_2, z_4) = 0,\ V_{i2}(z_2, z_4) = U_{i2} - W_i\mu_2(z_2),\ V_{i4}(z_2, z_4) = U_{i4} - \mu_4(z_4) \tag{3}$$

where $W_i > 0$ and takes on multiple values with positive probability. W_i parametrizes individual i's instrument sensitivity: individuals i with high W_i are relatively more responsive to changes in z_2 than changes in z_4. Note that this example therefore also does not satisfy Assumption ND: a discrete increase in both z_2 and z_4 could increase the relative utility from treatment status 2 for agents with high W_i while decreasing the relative utility for agents with low W_i, causing high W_i individuals to shift from 4 to 2 and low W_i individuals to shift from 2 to 4.

Proposition 2. *Suppose Assumptions UPM, TRUM.1, TRUM.2, and TRUM.3 hold. Then Equation 3 does not satisfy Assumption IC.*

Proof. I begin by, to the left side of the equality in Assumption IC, sequentially applying Bayes rule, substituting using Equation 3, and applying the law of iterated expectations.

$$\begin{aligned}
&\lim_{z_2'\uparrow z_2} \mathbf{P}\left[i \in I \middle| D_i(z_2', z_4) = 2, D_i(z_2, z_4) = 4\right] \\
&= \mathbf{P}[i \in I] \left(\lim_{z_2'\uparrow z_2} \frac{\mathbf{P}\left[D_i(z_2', z_4) = 2, D_i(z_2, z_4) = 4 \middle| i \in I\right]}{\mathbf{P}\left[D_i(z_2', z_4) = 2, D_i(z_2, z_4) = 4\right]} \right) \\
&= \mathbf{P}[i \in I] \left(\lim_{z_2'\uparrow z_2} \frac{\mathbf{P}\left[U_{i4} \geq \mu_4(z_4), U_{i2} \in [U_{i4} - \mu_4(z_4) + W_i\mu_2(z_2'), U_{i4} - \mu_4(z_4) + W_i\mu_2(z_2)] \middle| i \in I\right]}{\mathbf{P}\left[U_{i4} \geq \mu_4(z_4), U_{i2} \in [U_{i4} - \mu_4(z_4) + W_i\mu_2(z_2'), U_{i4} - \mu_4(z_4) + W_i\mu_2(z_2)]\right]} \right) \\
&= \mathbf{P}[i \in I] \left(\lim_{z_2'\uparrow z_2} \frac{\mathbf{E}\left[\mathbf{P}\left[U_{i4} \geq \mu_4(z_4), U_{i2} \in [U_{i4} - \mu_4(z_4) + W_i\mu_2(z_2'), U_{i4} - \mu_4(z_4) + W_i\mu_2(z_2)] \middle| i \in I, W_i\right]\right]}{\mathbf{E}\left[\mathbf{P}\left[U_{i4} \geq \mu_4(z_4), U_{i2} \in [U_{i4} - \mu_4(z_4) + W_i\mu_2(z_2'), U_{i4} - \mu_4(z_4) + W_i\mu_2(z_2)] \middle| W_i\right]\right]} \right)
\end{aligned}$$

The numerator and the denominator of the expression inside the limit both converge to zero; I therefore apply L'Hopital's rule to evaluate the limit, replacing the numerator and denominator with their derivative with respect to z_2' evaluated at $z_2' = z_2$. Letting $f(u_2, u_4|w)$ denote the density of (U_{i2}, U_{i4}), evaluated at (u_2, u_4), conditional on $W_i = w$, I note that

$$\begin{aligned}
&\frac{d}{dz_2'} \mathbf{P}\left[U_{i4} \geq \mu_4(z_4), U_{i2} \in [U_{i4} - \mu_4(z_4) + w\mu_2(z_2), U_{i4} - \mu_4(z_4) + w\mu_2(z_2')] \middle| W_i = w\right]\Big|_{z_2'=z_2} \\
&= \frac{d}{dz_2'} \int_{\mu_4(z_4)}^{\infty} \int_{u_4-\mu_4(z_4)+w\mu_2(z_2)}^{u_4-\mu_4(z_4)+w\mu_2(z_2')} f(u_2, u_4|w) du_2 du_4 \Bigg|_{z_2'=z_2} \\
&= w\mu_2'(z_2) \int_{\mu_4(z_4)}^{\infty} f(u_4 - \mu_4(z_4) + w\mu_2(z_2), u_4|w) du_4
\end{aligned}$$

I apply the same steps to the numerator, and include $i \in I$ in the conditional density

$f(u_2, u_4 | i \in I, w)$ to denote the additional conditioning on $i \in I$. Substituting yields

$$\lim_{z_2' \uparrow z_2} \mathbf{P}\left[i \in I | \, D_i(z_2', z_4) = 2, D_i(z_2, z_4) = 4\right]$$
$$= \mathbf{P}\left[i \in I\right] \frac{\mathbf{E}\left[W_i \int_{\mu_4(z_4)}^{\infty} f(u_4 - \mu_4(z_4) + W_i \mu_2(z_2), u_4 | i \in I, W_i) du_4\right]}{\mathbf{E}\left[W_i \int_{\mu_4(z_4)}^{\infty} f(u_4 - \mu_4(z_4) + W_i \mu_2(z_2), u_4 | W_i) du_4\right]} \tag{4}$$

Applying the steps above to the right side of the equality in Assumption IC yields

$$\lim_{z_4' \downarrow z_4} \mathbf{P}\left[i \in I | \, D_i(z_2, z_4') = 2, D_i(z_2, z_4) = 4\right]$$
$$= \mathbf{P}\left[i \in I\right] \frac{\mathbf{E}\left[\int_{\mu_4(z_4)}^{\infty} f(u_4 - \mu_4(z_4) + W_i \mu_2(z_2), u_4 | i \in I, W_i) du_4\right]}{\mathbf{E}\left[\int_{\mu_4(z_4)}^{\infty} f(u_4 - \mu_4(z_4) + W_i \mu_2(z_2), u_4 | W_i) du_4\right]} \tag{5}$$

Equations 4 and 5 imply that the Assumption IC does not generically hold under Equation 3; the limit as $z_2' \uparrow z_2$ weighs the density of marginal individuals with $i \in \mathcal{I}$ proportionally to sensitivity to z_2 (high W_i) while the limit as $z_4' \downarrow z_4$ does not. $\square$

When W_i is heterogeneous, Proposition 2 establishes that Equation 3 need not satisfy any of Assumptions ARUM, ND, and IC. Alternatively, when W_i is constant, Equation 3 satisfies Assumptions ARUM, ND, and IC. This sharpness suggests the equivalence of these three assumptions.

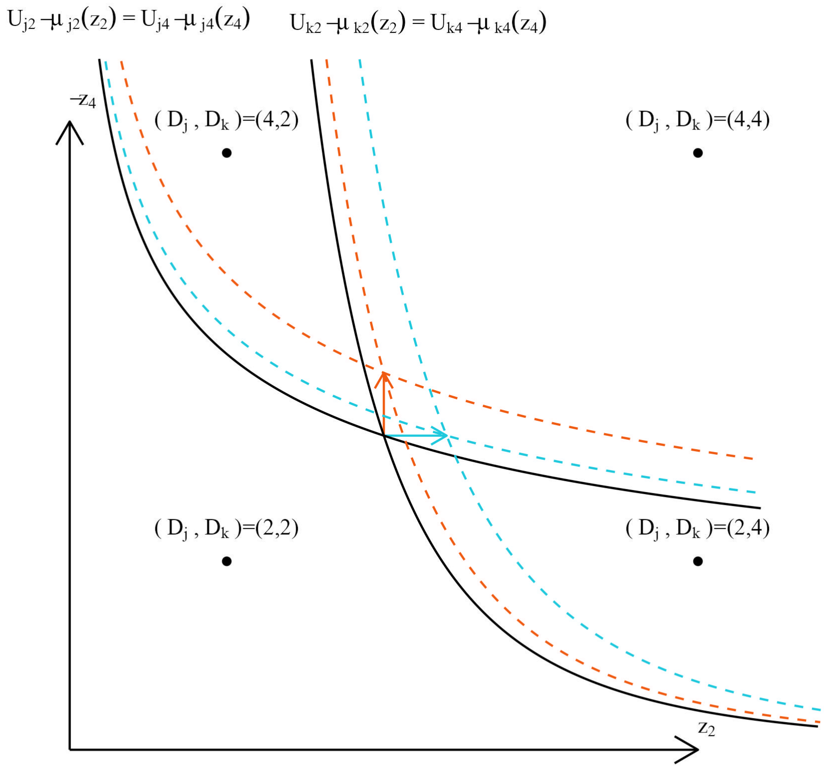
2.2 Proof of equivalence

Proposition 3. *Suppose Assumptions UPM, TRUM.1, TRUM.2, and TRUM.3 hold. Then Assumptions IC, ND, and ARUM are equivalent.*

Proof. To begin, I note that Assumption ARUM implies Assumption IC (as shown by Mountjoy, 2022) and Assumption ND (as shown by Lee & Salanié, 2023). Below, I show that either Assumption IC or Assumption ND implies homogeneous instrument sensitivity, which I show implies Assumption ARUM. Therefore, Assumptions IC, ND, and ARUM are equivalent.

The intuition underlying the result that either Assumption IC or Assumption ND is equivalent to homogeneous instrument sensitivity, and therefore Assumption ARUM, is present in Figure 3. Figure 3 builds closely on Mogstad et al. (2021), applying their Figure 2 (and associated logic) to the setting with multivalued treatment

Figure 3: Identical compliers or no defiers implies homogeneous instrument sensitivity



Notes: Two example indifference curves, corresponding to the set of values z for which individuals of type-j and type-k are indifferent between treatment statuses 2 and 4, are plotted in this graph, which is centered at their point of intersection. Values of the treatment status of type-j and type-k individuals (D_j, D_k) which contradict Assumption ND are plotted. Indifference curves through alternative values of z − with an increase in z_2 and with a decrease in z_4 − are plotted for both individuals of type-μ_j and type-μ_k, which contradict Assumption IC.

and extending it from Assumption ND to Assumption IC. First, Figure 3 presents an example with two defiers j and k for a given change in the instrument. Under Assumption UPM, the implied sets of (z_2, z_4) for which defiers j and k are indifferent between treatment statuses must intersect, that is j and k have heterogeneous instrument sensitivity. One can similarly identify defiers from j-type and k-type individuals who differ in their instrument sensitivity. Second, Figure 3 shows the effects of a small increase in z_2 and a small decrease in z_4 on individuals with instrument sensitivity characterized by (μ_{j2}, μ_{j4}) and (μ_{k2}, μ_{k4}); more of the μ_k-type individuals shift treatment status in response to the increase in z_2, while more of the μ_j-type individuals shift treatment status in response to the decrease in z_4.

Homogeneous instrument sensitivity $\Rightarrow$ Assumption ARUM I define homogeneous instrument sensitivity as

$$\frac{\mu'_{j2}(z_2)}{\mu'_{k2}(z_2)} = \frac{\mu'_{j4}(z_4)}{\mu'_{k4}(z_4)} \quad \forall j,k \in \mathcal{I}, (z_2, z_4) \in \mathcal{Z}$$

These ratios are well defined under Assumption TRUM.3. As $\mathcal{Z}$ is a 2-dimensional interval, if $(z'_2, z'_4) \in \mathcal{Z}$ then $(z_2, z'_4), (z'_2, z_4) \in \mathcal{Z}$, and the above equality implies

$$\frac{\mu'_{j2}(z_2)}{\mu'_{k2}(z_2)} = \frac{\mu'_{j2}(z'_2)}{\mu'_{k2}(z'_2)} = \frac{\mu'_{j4}(z_4)}{\mu'_{k4}(z_4)} = \frac{\mu'_{j4}(z'_4)}{\mu'_{k4}(z'_4)} \quad \forall j,k \in \mathcal{I}, (z_2, z_4), (z'_2, z'_4) \in \mathcal{Z}$$

Fix any $\ell \in \mathcal{I}$. By continuous differentiability of $\mu_{i2}(z_2)$ and $\mu_{i4}(z_4)$ by Assumption TRUM.1, rescaling $V_{id}(z)$ implied by Assumption UPM in Proposition 1 by the i-specific constant $\frac{\mu'_{\ell 4}(z_4)}{\mu'_{i4}(z_4)} = \frac{\mu'_{\ell 2}(z_2)}{\mu'_{i2}(z_2)}$ yields the additive random utility model in Assumption ARUM with $\mu_2(z) = -\mu_{\ell 2}(z_2)$ and $\mu_4(z) = -\mu_{\ell 4}(z_4)$.

Assumption IC $\Rightarrow$ Homogeneous instrument sensitivity I apply the steps in the proof of Proposition 2, but conditioning on (μ_{i2}, μ_{i4}) instead of W_i. This yields

$$\begin{aligned} &\lim_{z'_2 \uparrow z_2} \mathbf{P}\left[i \in I \mid D_i(z'_2, z_4) = 2, D_i(z_2, z_4) = 4\right] \\ &\quad = \mathbf{P}\left[i \in I\right] \frac{\mathbf{E}\left[\mu'_{i2}(z_2) \int_{\mu_{i4}(z_4)}^{\infty} f(u_4 - \mu_{i4}(z_4) + \mu_{i2}(z_2), u_4 | i \in I, \mu_{i2}, \mu_{i4}) du_4\right]}{\mathbf{E}\left[\mu'_{i2}(z_2) \int_{\mu_{i4}(z_4)}^{\infty} f(u_4 - \mu_{i4}(z_4) + \mu_{i2}(z_2), u_4 | \mu_{i2}, \mu_{i4}) du_4\right]} \end{aligned} \tag{6}$$

$$\begin{aligned} &\lim_{z'_4 \downarrow z_4} \mathbf{P}\left[i \in I \mid D_i(z_2, z'_4) = 2, D_i(z_2, z_4) = 4\right] \\ &\quad = \mathbf{P}\left[i \in I\right] \frac{\mathbf{E}\left[\mu'_{i4}(z_4) \int_{\mu_{i4}(z_4)}^{\infty} f(u_4 - \mu_{i4}(z_4) + \mu_{i2}(z_2), u_4 | i \in I, \mu_{i2}, \mu_{i4}) du_4\right]}{\mathbf{E}\left[\mu'_{i4}(z_4) \int_{\mu_{i4}(z_4)}^{\infty} f(u_4 - \mu_{i4}(z_4) + \mu_{i2}(z_2), u_4 | \mu_{i2}, \mu_{i4}) du_4\right]} \end{aligned} \tag{7}$$

This derivation requires Assumption TRUM.1 (for differentiability of $\mu_{i2}(z_2)$ and $\mu_{i4}(z_4)$) and Assumption TRUM.2 (for the application of limits and for conditional densities to be well-defined). Assumption IC implies Equations 6 and 7 are equal; these equations are integrals of the density of "marginal" individuals, with "marginal" individual i weighted by $\mu'_{i2}(z_2)$ and $\mu'_{i4}(z_4)$. Equality for all possible $I \subseteq \mathcal{I}$ and $(z_2, z_4) \in \mathcal{Z}$ requires these weighting schemes are identical, and therefore $\frac{\mu'_{j2}(z_2)}{\mu'_{j4}(z_4)} = \frac{\mu'_{k2}(z_2)}{\mu'_{k4}(z_4)}$ for all $j, k \in \mathcal{I}$, $(z_2, z_4) \in \mathcal{Z}$. This is equivalent to homogeneous instrument sensitivity.

Assumption ND $\Rightarrow$ Homogeneous instrument sensitivity By Assumptions TRUM.1, TRUM.2, and TRUM.3, for each type (μ_{i2}, μ_{i4}) there are a positive density of marginal individuals at any value of the instrument (z_2, z_4). Assumption TRUM.1 holds, and Proposition 2 of Mogstad et al. (2021) therefore implies that

$$\mu'_{j2}(z_2)\mu'_{k4}(z_4) = \mu'_{k2}(z_2)\mu'_{j4}(z_4)$$

For all $(j, k) \in \mathcal{I}$, $(z_2, z_4) \in \mathcal{Z}$. By Assumption TRUM.2, this is equivalent to homogeneous instrument sensitivity. $\square$

3 Identification and assumptions on selection

The conditional equivalence of Assumptions IC, ND, and ARUM immediately implies that any results derived under one of these assumptions immediately hold under the others. I discuss implications of this equivalence below for identification in Section 3.1 and for falsifiability in Section 3.2.

3.1 Identification and assumptions on selection

I summarize two distinct approaches to local identification of treatment effects: one based on Assumption IC developed by Mountjoy (2022), and one I derive from Assumption ARUM.

I apply notation from Assumption ARUM, with heterogeneity in $D_i(z)$ fully characterized by (U_{i2}, U_{i4}). I let Y_{id} be the outcome of individual i under treatment status d, such that the observed outcome $Y_i(Z_i) \equiv \sum_{d \in \{0,2,4\}} \mathbf{1}\{D_i(Z_i) = d\} Y_{id}$. The econometrician observes $(Y_i(Z_i), D_i(Z_i), Z_i)$ for each individual $i \in \mathcal{I}$.

I additionally make the following independence assumption.

Assumption IE (Independence and Exclusion)**.** $Z_i \perp (Y_{i0}, Y_{i2}, Y_{i4}, U_{i2}, U_{i4})$

Assumption IE implies that the assigned value of the instrument Z_i is independent of potential outcomes and selection into treatment.

Identification derived under Assumption IC I restate Equation 14 of Mountjoy (2022) in Equation 8; it establishes that the average potential outcome under treatment statuses 2 among individuals indifferent between treatment statuses 2 and 4 is identified under Assumptions UPM, IC, and IE.

$$\mathbf{E}\left[Y_{i2}|U_{i2} - \mu_2(z_2) = U_{i4} - \mu_4(z_4) > 0\right] = \frac{\frac{d}{dz_4}\mathbf{E}\left[Y_i(Z_i)D_{i2}(Z_i)|Z_i = (z_2, z_4)\right]}{\frac{d}{dz_4}\mathbf{E}\left[D_{i2}(Z_i)|Z_i = (z_2, z_4)\right]} \tag{8}$$

Roughly, any increase in treatment status 2 caused by an increase in z_4 is from individuals who are indifferent between treatment status 2 and treatment status 4. Therefore, the excess mass of outcome among individuals with treatment status 2 must be from those marginal compliers, and the average outcome among those marginal compliers from treatment status 2 to treatment status 4 is identified.

Mountjoy (2022) shows that average potential outcomes under alternative treatment statuses d, among individuals indifferent between treatment statuses d and $d' \neq d$, are identified using a related approach; the average treatment effect for individuals indifferent between any pair of treatments is therefore identified.

Identification derived under Assumption ARUM Under Assumption ARUM, other conditional expectations are locally identified. In Equations 9 and 10, I establish identification of the average potential outcome under treatment status 2 among individuals indifferent between all three treatment statuses, under Assumptions UPM, ARUM, and IE. Let $U_i = \mu(z)$ denote that $(U_{i2}, U_{i4}) = (\mu_2(z_2), \mu_4(z_4))$, that is that individual i is indifferent between all three treatment statuses; then,

$$\mathbf{E}\left[Y_{i2}|U_i = \mu(z)\right] = \frac{\left(\frac{d^2}{dz_4^2} + \frac{\mu_4'(z_4)}{\mu_2'(z_2)}\frac{d^2}{dz_2 dz_4}\right)\mathbf{E}\left[Y_i(Z_i)D_{i2}(Z_i)|Z_i = (z_2, z_4)\right]}{\left(\frac{d^2}{dz_4^2} + \frac{\mu_4'(z_4)}{\mu_2'(z_2)}\frac{d^2}{dz_2 dz_4}\right)\mathbf{E}\left[D_{i2}(Z_i)|Z_i = (z_2, z_4)\right]} \tag{9}$$

$$\frac{\mu_4'(z_4)}{\mu_2'(z_2)} = \frac{\frac{d}{dz_4}\mathbf{E}\left[D_{i2}(Z_i)|Z_i = (z_2, z_4)\right]}{\frac{d}{dz_2}\mathbf{E}\left[D_{i4}(Z_i)|Z_i = (z_2, z_4)\right]} \tag{10}$$

While Equation 9 is a direct application of Theorem 3.1 of Lee & Salanié (2018), Lee & Salanié (2018) note that there is not generic guidance for local identification of utility indices. Equation 10 was derived by Allen & Rehbeck (2019) and Bhattacharya (2023), who establish local identification of utility indices up to location and scale normalizations under Assumptions UPM, ARUM, and IE.

Equation 10 implies that relative instrument sensitivity is identified. To see this, first note that the mass of compliers from treatment status 4 to treatment status 2 caused by a small decrease in z_2 is proportional to the density of individuals indifferent between treatment status 2 and 4 times instrument sensitivity $\mu_2'(z_2)$. Similarly, the mass of compliers from treatment status 2 to treatment status 4 caused by a small decrease in z_4 is proportional to the same density of indifferent individuals times instrument sensitivity $\mu_4'(z_4)$; relative instrument sensitivity $\frac{\mu_4'(z_4)}{\mu_2'(z_2)}$ is therefore equal to this ratio of complier densities.

Once relative instrument sensitivities are identified, Equation 9 implies the mass of individuals who are indifferent between all three treatment statuses can be isolated, as presented in Figure 4a. Increasing z_4 pushes individuals indifferent between treatment status 2 and 4 into treatment status 2, decreasing equally both $\mu_2(z_2)$ and $\mu_4(z_4)$ pushes individuals indifferent between treatment status 2 and 0 into treatment status 2, and doing both of these also pushes individuals indifferent between all three treatment statuses into treatment status 2; this suggests the ratio of the "local difference-in-differences" in Equation 9.

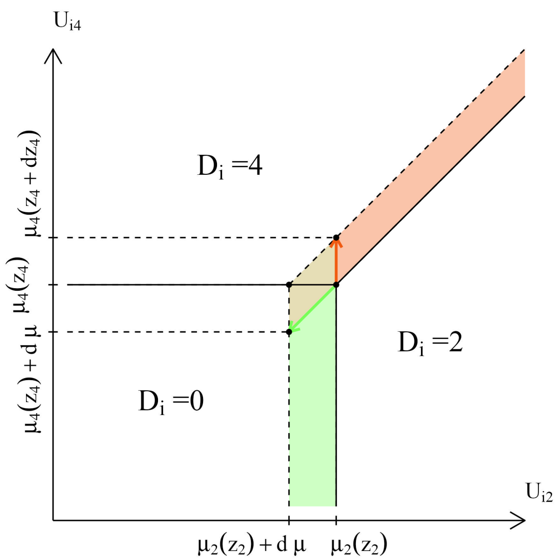
The average potential outcome under treatment 0 or treatment 4 among individuals indifferent between all three treatments is identified in a similar manner to the average potential outcome under treatment 2, respectively presented in Figures 4b and 4c. Importantly, this implies local identification of the effect of both treatment 2 and treatment 4 for a single group of individuals, those indifferent between all three treatment statuses.

3.2 Tests of assumptions on selection

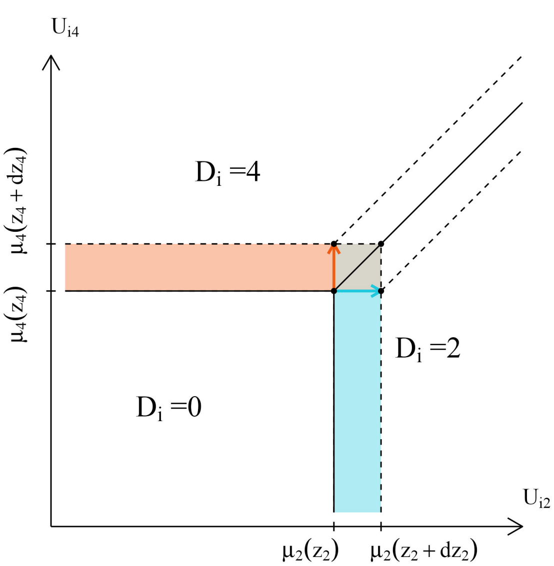
Tests derived under Assumption IC Mountjoy (2022) proposes a test of Assumption IC: average characteristics of individuals induced to shift from treatment 2 to treatment 4 in response to a decrease in z_4 and individuals induced to shift from treatment 4 to treatment 2 in response to a decrease in z_2 are the same.

Figure 4: Local difference-in-differences identifies treatment effects for individuals indifferent across all treatment statuses

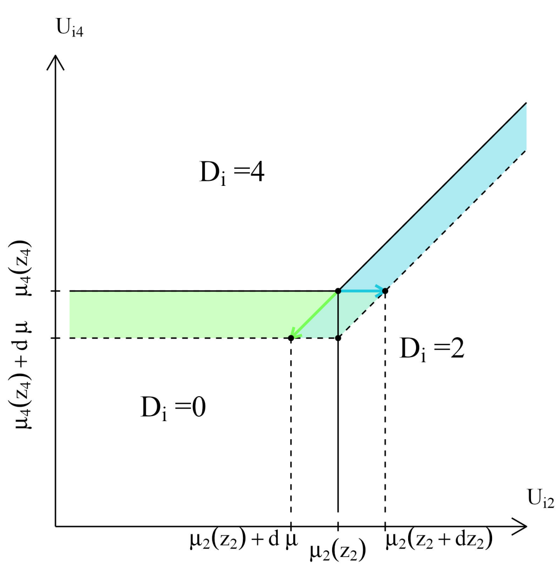
(a) Identification of $\mathbf{E}\left[Y_{i2}|U_i = \mu(z)\right]$



(b) Identification of $\mathbf{E}\left[Y_{i0}|U_i = \mu(z)\right]$



(c) Identification of $\mathbf{E}\left[Y_{i4}|U_i = \mu(z)\right]$



Notes: The set of individuals U_i with treatment status 0, 2, and 4 under small changes in the value of the instruments z are plotted in each panel of this Figure. Shaded regions in panels (a), (b), and (c) correspond to changes in the set of individuals with treatment status 2, 0, and 4, respectively, under each of two possible changes in the values of the instruments or both.

Tests derived under Assumption ND Many tests of Assumption ND have been developed in the context of testing monotonicity with a binary instrument (Heckman & Vytlacil, 2005; Kitagawa, 2015; Rose & Shem-Tov, 2023). In general, these tests build on the following intuition: if a change in the value of the instrument $z \to z'$ induces individuals to shift into treatment d, but not out of treatment d, then the probability of observing individuals who both adopt treatment d and have any characteristic or outcome should increase. Otherwise, either some individuals must have shifted out of treatment d, or the instrument must have changed individuals' outcomes without changing their treatment status (a violation of Assumption IE).

Tests derived under Assumption ARUM Allen & Rehbeck (2019) shows that Assumption ARUM implies Equation 10 holding for all $z \in \mathcal{Z}$, while Bhattacharya (2023) further establishes equivalence. This implies a testable restriction on the impacts of the instruments on treatment probabilities at any set of instrument values $\{(z_2, z_4), (z_2, z_4'), (z_2', z_4), (z_2', z_4')\}$.

The equivalence of Assumptions ND and ARUM implies that Equation 10 is a testable restriction of Assumption ND on the impacts of the instruments on treatment probabilities conditional on Assumption UPM. The equivalence of Equation 10 and Assumption ARUM conditional on Assumption UPM established in Bhattacharya (2023) further implies that Equation 10 is the *only* testable restriction of Assumptions ND and IC on treatment probabilities conditional on Assumption UPM.

This testable restriction of ARUM contrasts with the tests of Assumption ND described above, which make use of observed outcomes or exogenous characteristics. The existence of this test is perhaps surprising, as Assumption ND is not testable from choice probabilities alone with a binary instrument, and highlights the identifying power of Assumption UPM.

4 Conclusion

Identification of effects of multiple treatments with instrumental variables brings challenges not present in the case of a binary treatment (Imbens & Angrist, 1994): monotonicity and instrument independence and exclusion are no longer sufficient to identify even average flows of individuals between treatments. Recent work has derived identification of treatment effects under parsimonious and testable generalizations of the monotonicity assumption on choice behavior: unordered partial monotonicity and either identical compliers (Mountjoy, 2022) or an additive random utility model (Lee & Salanié, 2018). Analogously to Vytlacil (2002), I show that these assumptions are equivalent to no defiers; this equivalence immediately implies that approaches to identification and falsification derived under one assumption are valid under others.

The novel results in this paper on identification and falsifiability under no defiers (or, equivalently, identical compliers or the additive random utility model) suggest that no defiers may be a natural starting assumption to analyze the effects of unordered treatments under unordered partial monotonicity:

- Unordered partial monotonicity is reasonable in many contexts: when instruments are prices, it is implied by the random utility model.
- No defiers is equivalent to a testable assumption on choice probabilities with continuous variation in the instruments for all treatments.
- All treatment effects are identified for individuals indifferent between all treatments within the range of this variation.

This paper does not develop an estimator or a statistical test. Conditional on estimation of additive utility indices and the distribution of unobserved heterogeneity, results from Kline & Walters (2016) suggest control function approaches can be used to recover treatment effects. Developing statistical tests of the additive random utility model, and developing estimators of additive utility indices and the distribution of unobserved heterogeneity without parametric restrictions on their functional forms, are important directions for future work.

The conditional equivalence of no defiers, identical compliers, and the additive random utility model underscores the difficulty of identification of treatment effects under more general restrictions. Absent these restrictions on heterogeneity of individual treatment responses, unordered partial monotonicity and instrument independence and exclusion are sufficient to bound treatment effects with discrete instruments (Kamat et al., 2023; Lee & Salanié, 2023), and to identify treatment effects with large instrument support (Heckman et al., 2008). As alternatives, assumptions on selection on outcomes, or weaker restrictions on heterogeneous instrument sensitivity, may enable tighter bounds on treatment effects while retaining falsifiability.

References

Allen, R. & Rehbeck, J. (2019). Identification with additively separable heterogeneity. *Econometrica*, 87(3), 1021–1054.

Behaghel, L., Crépon, B., & Gurgand, M. (2013). Robustness of the encouragement design in a two-treatment randomized control trial.

Bhattacharya, D. (2023). Integrability and identification in multinomial choice models.

Bhuller, M. & Sigstad, H. (2023). 2sls with multiple treatments.

Heckman, J. J. & Pinto, R. (2018). Unordered monotonicity. *Econometrica*, 86(1), 1–35.

Heckman, J. J., Urzua, S., & Vytlacil, E. (2008). Instrumental variables in models with multiple outcomes: The general unordered case. *Annals of Economics and Statistics*, (pp. 151–174).

Heckman, J. J. & Vytlacil, E. (2005). Structural equations, treatment effects, and econometric policy evaluation. *Econometrica*, 73(3), 669–738.

Imbens, G. W. & Angrist, J. D. (1994). Identification and estimation of local average treatment effects. *Econometrica (1986-1998)*, 62(2), 467.

Kamat, V., Norris, S., & Pecenco, M. (2023). Identification in multiple treatment models under discrete variation.

Kirkeboen, L. J., Leuven, E., & Mogstad, M. (2016). Field of study, earnings, and self-selection. *The Quarterly Journal of Economics*, 131(3), 1057–1111.

Kitagawa, T. (2015). A test for instrument validity. *Econometrica*, 83(5), 2043–2063.

Kline, P. & Walters, C. R. (2016). Evaluating public programs with close substitutes: The case of head start. *The Quarterly Journal of Economics*, 131(4), 1795–1848.

Lee, S. & Salanié, B. (2018). Identifying effects of multivalued treatments. *Econometrica*, 86(6), 1939–1963.

Lee, S. & Salanié, B. (2023). Filtered and unfiltered treatment effects with targeting instruments.

Mogstad, M., Torgovitsky, A., & Walters, C. R. (2021). The causal interpretation of two-stage least squares with multiple instrumental variables. *American Economic Review*, 111(11), 3663–3698.

Mountjoy, J. (2022). Community colleges and upward mobility. *American Economic Review*, 112(8), 2580–2630.

Navjeevan, M. & Pinto, R. (2022). Ordered, unordered and minimal monotonicity criteria.

Rose, E. K. & Shem-Tov, Y. (2023). On recoding ordered treatments as binary indicators. *arXiv preprint arXiv:2111.12258*.

Vytlacil, E. (2002). Independence, monotonicity, and latent index models: An equivalence result. *Econometrica*, 70(1), 331–341.

Appendix

Alternative proof of Assumption UPM $\Rightarrow$ Assumption TRUM. Fix i such that Assumption UPM holds, and suppose there exists $z, z', z'' \in \mathcal{Z}$ such that $D_i(z) = 0, D_i(z') = 2, D_i(z'') = 4$; I show that Assumption TRUM holds. It follows that Assumption UPM implies Assumption TRUM. To show this:

- I begin by showing there exists z^*_{i2}, z^*_{i4} such that $D_i(z) = 0$ if and only if $z_2 \geq z^*_{i2}$ and $z_4 \geq z^*_{i4}$, $D_i(z) = 2$ only if $z_2 < z^*_{i2}$, and $D_i(z) = 4$ only if $z_4 < z^*_{i4}$. This is analogous to the analysis of deviations A and B in Figure 1.
- The above fully characterizes $D_i(z)$ in three of four quadrants of $\mathcal{Z}$ relative to (z^*_{i2}, z^*_{i4}); for $z_2 < z^*_{i2}$ and $z_4 < z^*_{i4}$, I then show there exists $\overline{z}_{i4}(z_2)$ such that $\overline{z}_{i4}(z^*_{i2}) = z^*_{i4}$, $\overline{z}_{i4}(z_2)$ is increasing in z_2, and $D_i(z) = 4$ if and only if $z_4 < \overline{z}_{i4}(z_2)$ for $z_2 < z^*_{i2}$ and $z_4 < z^*_{i4}$. This is analogous to the analysis of deviation C in Figure 1.
- I then construct $\mu_{i2}(z_2)$ and $\mu_{i4}(z_4)$ satisfying Assumption TRUM from z^*_{i2}, z^*_{i4}, and $\overline{z}_{i4}(z_2)$.

Construction of z^*_{i2} and z^*_{i4} Let

$$z^*_{i2} \equiv \max_{z \in \mathcal{Z}, D_i(z)=2} z_2 \qquad z^*_{i4} \equiv \max_{z \in \mathcal{Z}, D_i(z)=4} z_4$$

By construction, $D_i(z) = 2$ only if $z_2 < z^*_{i2}$ and $D_i(z) = 4$ only if $z_4 < z^*_{i4}$. As a consequence, $D_i(z) = 0$ if $z_2 < z^*_{i2}$ and $z_4 < z^*_{i4}$. It remains to show that $D_i(z) = 0$ only if $z_2 \geq z^*_{i2}$ and $z_4 \geq z_{i4}*$.

Suppose, for contradiction, there exists $z \equiv (z_2, z_4)$ such that $z_2 < z^*_{i2}$ and $D_i(z) = 0$. By definition of z^*_{i2}, there exists $z' \equiv (z'_2, z'_4)$ such that $z'_2 > z_2$ and $D_i(z') = 2$. By Assumption UPM, $D_i((z'_2, z_4)) = 0$, since decreasing z'_2 to z_2 cannot shift i $4 \to 0$ nor $2 \to 0$. This implies $D_i((z'_2, z'_4)) = 2$ and $D_i((z'_2, z_4)) = 0$, which contradicts Assumption UPM. So there does not exist $z_2 < z^*_{i2}$ such that $D_i(z) = 0$. Symmetrically, there does not exist $z_4 < z^*_{i4}$ such that $D_i(z) = 0$. Therefore, $D_i(z) = 0$ only if $z_2 \geq z^*_{i2}$ and $z_4 \geq z^*_{i4}$.

Construction of $\overline{z}_{i4}(z_2)$ Let

$$\overline{z}_{i4}(z_2) \equiv \begin{cases} \max\limits_{(z_2,z_4)\in\mathcal{Z}} z_4 & z_2 < z_{i2}^* \\ z_{i4}^* & z_2 \geq z_{i2}^* \end{cases}$$

By construction, for $z_2 < z_{i2}^*$ and $z_4 < z_{i4}^*$, $D_i(z) = 4$ only if $z_4 < \overline{z}_{i4}(z_2)$. It remains to show that $D_i(z) = 4$ if $z_4 < \overline{z}_{i4}(z_2)$, and that $\overline{z}_{i4}(z_2)$ is increasing in z_2.

First, suppose for contradiction, there exists $z = (z_2, z_4)$ such that $z_2 < z_{i2}^*$, $z_4 < \overline{z}_{i4}(z_2)$, and $D_i(z) = 2$. By definition of $\overline{z}_{i4}(z_2)$, there exists $z_4' > z_4$ such that $D_i((z_2, z_4')) = 4$. This contradicts Assumption UPM, so $D_i(z) = 4$ if $z_4 < \overline{z}_{i4}(z_2)$.

Second, suppose for contradiction that $\overline{z}_{i4}(z_2)$ is not increasing. Note that it must then not be increasing for $z_2 \leq z_{i2}^*$, as by construction $\overline{z}_{i4}(z_2) \leq z_{i4}^*$. There must then exist z_2, z_2' such that $z_2 < z_2' < z_{i2}^*$ and $\overline{z}_{i4}(z_2) > \overline{z}_{i4}(z_2')$. There must then exist $z = (z_2, z_4)$, $z' = (z_2', z_4')$ such that $\overline{z}_{i4}(z_2) \geq z_4 > z_4' > \overline{z}_{i4}(z_2')$, and $D_i(z) = 4$ and $D_i(z') = 2$. Note that $D_i((z_2', z_4)) \neq 0$ as $z_2' < z_{i2}^*$, $D_i((z_2', z_4)) \neq 2$ by Assumption UPM as $D_i((z_2, z_4)) = 4$ and $z_2' > z_2$, and $D_i((z_2', z_4)) \neq 4$ by Assumption UPM as $D_i((z_2', z_4')) = 2$ and $z_4 > z_4'$. This yields a contradiction, so $\overline{z}_{i4}(z_2)$ must be increasing.

Construction of $\mu_{i2}(z_2)$ and $\mu_{i4}(z_4)$ satisfying Assumption TRUM Let

$$\begin{aligned} \mu_{i2}(z_2) &\equiv \mathbf{1}\{z_2 > z_{i2}^*\}(z_2 - z_{i2}^*) + \mathbf{1}\{z_2 \leq z_{i2}^*\}(\overline{z}_{i4}(z_2) - z_{i4}^*) \\ \mu_{i4}(z_4) &\equiv z_4 - z_{i4}^* \end{aligned}$$

and $U_{i2} = U_{i4} = 0$, so $V_{i2}(z) = -\mu_{i2}(z_2)$ and $V_{i4}(z) = -\mu_{i4}(z_4)$. Below I establish that this construction satisfies Assumption TRUM.

First, note that $\mu_{i2}(z_2) > 0$ if and only if $z_2 > z_{i2}^*$. If $z_2 > z_{i2}^*$, then $\mu_{i2}(z_2) = z_2 - z_{i2}^* > 0$. If $z_2 \leq z_{i2}^*$, then $\mu_{i2}(z_2) = \overline{z}_{i4}(z_2) - z_{i4}^* \leq 0$, since $\overline{z}_{i4}(z_2)$ is increasing in z_2 and $\overline{z}_{i4}(z_{i2}^*) = z_{i4}^*$. Similarly, note that $\mu_{i4}(z_4) \equiv z_4 - z_{i4}^* > 0$ if and only if $z_4 > z_{i4}^*$. As a consequence:

- If $z_2 > z_{i2}^*$ and $z_4 > z_{i4}^*$ ($D_i(z) = 0$), then $\mu_{i2}(z_2) > 0$ and $\mu_{i4}(z_4) > 0$.
- If $z_2 < z_{i2}^*$ and $z_4 > z_{i4}^*$ ($D_i(z) = 2$), then $\mu_{i2}(z_2) < 0$ and $\mu_{i4}(z_4) > 0$.
- If $z_2 > z_{i2}^*$ and $z_4 < z_{i4}^*$ ($D_i(z) = 4$), then $\mu_{i2}(z_2) > 0$ and $\mu_{i4}(z_4) < 0$.

Second, for $z_2 < z_{i2}^*$ and $z_4 < z_{i4}^*$, $\mu_{i2}(z_2) > \mu_{i4}(z_4)$ if and only if $\overline{z}_{i4}(z_2) > z_4$.

Therefore, $D_i(z) = 2$ if $\mu_{i2}(z_2) > \mu_{i4}(z_4)$ and $D_i(z) = 4$ if $\mu_{i2}(z_2) < \mu_{i4}(z_4)$.

This characterization of $D_i(z)$ from $\mu_{i2}(z_2)$ and $\mu_{i4}(z_4)$ implies that $D_i(z) = \arg\max_{d \in \{0,2,4\}} V_{id}(z)$, and Assumption TRUM holds.

$\square$